LAMPLIGHTERS GROUPS
THEIR PRESENTATIONS, CAYLELY GRAPHS, AND GENERATING AUTOMATA

by

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A Honors Thesis
Submitted to the
Faculty
of
George Mason University
In Partial fulfillment of
The Requirements for the Honors Degree
of
B.S.
Mathematics

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Date: ______________________ Spring Semester 2023
George Mason University
Fairfax, VA

Acknowledgments

I'd like to thank Dr. Lukyanenko for his agreement to support what was intended to be a thesis on analytic and geometric topics, and I'd especially like to thank him for his continued support when it became clear that the contents of the thesis were to be primarily algebraic. I'd also like to thank Julian Benali for an extremely helpful conversation pertaining to wreath products, as well as Violet Nguyen for her friendship and frequent Tex support, which helped keep me sane while working on this project throughout the past year.

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Abstract

LAMPLIGHTERS GROUPS

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The focus of this paper is the traditional lamplighter group $L_2 = (\mathbb{Z}/2\mathbb{Z}) \operatorname{wr} \mathbb{Z}$ and the broader class of lamplighter groups $L_G = G \operatorname{wr} \mathbb{Z}$, where G is a finite group. These groups have seen direct use in papers such as [7], in which Grigorchuk, Linnell, Schick, and Żuk show there is a closed Riemannian manifold (with the HNN-extension of L_2 as its fundamental group) that has a rational L^2-Betti number. Other results rely on generalizing or building atop the lamplighter group, as is done by Brieussel and Zheng in [3] to expand the known possible speed, entropy, isoperimetric profile, return probability, and compression gap attainable by random walks on finitely generated groups.

This paper provides the algebraic background needed to understand the definition of the traditional lamplighter group, as well as its presentation, Caylely graph, and representation as an automaton group. All three of these constructions provide distinct avenues to generalize the lamplighter group. The sort of generalization discussed in this paper—moving from L_2 to L_G—is quite natural from the perspective of the lamplighter group's Cayley graph and presentation. The case of finding a suitable automaton to generate such generalizations is more interesting; only some lamplighter groups L_G can be generated by automata. When G is a finite Abelian group, then L_G can be generated by the *Cayley machine* $\mathcal{C}(G)$ of G,

and when G is a finite Non-Abelian group there is no automaton which can generate the group. These Cayley machines are interesting in their own right, so we explore the more general structure of the groups generated by Cayley machines.

Chapter 1: Introduction

This paper examines the *lamplighter group* $L_2 = (\mathbb{Z}/2\mathbb{Z}) \operatorname{wr} \mathbb{Z}$ and several of its generalizations. The name "lamplighter" derives from viewing an element of the group as an infinite street with integer addresses. There is a lamp installed at every address on the street. Only finitely many of these lamps are on, and there is a lonely inhabitant on the street: a lamplighter who stands by some particular lamppost. Under this analogy, the group is generated by two elements: one which causes the lamplighter to walk forward a block from their current position, and one which causes the lamplighter to interact with the lamp they are standing by.

The analogy lends itself well to generalizations. The street could install lights with varying brightness ($(\mathbb{Z}/n\mathbb{Z}) \operatorname{wr} \mathbb{Z}$), multi-color lights of varying brightness ($((\mathbb{Z}/n_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/n_m\mathbb{Z})) \operatorname{wr} \mathbb{Z}$), or some very elaborate lights that almost no one knows how to work ($G \operatorname{wr} \mathbb{Z}$ where G is finite non-Abelian).

In general, an element of L_2 is an integer-indexed binary sequence paired with an integer tag. Suppose we take two elements $((\ldots, s_{-1}, s_0, s_1, \ldots), M) = ((s_i)_{i\in\mathbb{Z}}, M)$ and $((\ldots, t_{-1}, t_0, t_1, \ldots), N) = ((t_i)_{i\in\mathbb{Z}}, N)$ from L_2. Then the product $((s_i)_{i\in\mathbb{Z}}, M) \cdot ((t_i)_{i\in\mathbb{Z}}, N)$ is taken as follows:

1. Shift the elements of the t-sequence right by M places, so that $(\ldots, t_{-1}, t_0, t_1, \ldots) = (t_i)_{i\in\mathbb{Z}}$ becomes $(\ldots, t_{-M-1}, t_{-M}, t_{-M+1}, \ldots) = (t_{i-M})_{i\in\mathbb{Z}}$.

2. Add the elements of the s-sequence and shifted t-sequence modulo 2, to get the sequence $(\ldots, s_{-1} + t_{-M-1}, s_0 + t_{-M}, s_1 + t_{-M+1}, \ldots) = (s_i + t_{i-M})_{i\in\mathbb{Z}}$.

3. Tag the resulting sequence with the sum of the two tags $M + N$, giving the element $((\ldots, s_{-1} + t_{-M-1}, s_0 + t_{-M}, s_1 + t_{-M+1}, \ldots), M + N) = ((s_i + t_{i-M})_{i\in\mathbb{Z}}, M + N)$.

To give a concrete example, suppose that $(\dots, s_{-1}, s_0, s_1, \dots) = (\dots, 1, 0, 1, \dots)$ and $M = 1$, and that $(\dots, t_{-1}, t_0, t_1, \dots) = (\dots, 0, 1, 1, \dots)$ and $N = -8$ (with all s_i and t_i not shown vanishing). Table 1 illustrates the multiplication process.

	Element Tag	Value at Index	$\cdots$	-2	-1	0	1	2	$\cdots$
Operand 1	1		$\cdots$	0	1	0	1	0	$\cdots$
Operand 2	-8		$\cdots$	0	0	1	1	0	$\cdots$
Shifted Sequence			$\cdots$	0	0	0	1	1	$\cdots$
Summed Sequences			$\cdots$	0	1	0	0	1	$\cdots$
Result	-7		$\cdots$	0	1	0	0	1	$\cdots$

Table 1.1: Multiplication of two lamplighter elements

Recent interest in Lamplighter groups stems from a few sources. In [8] Grigorchuk and Żuk proved that L_2 is generated by the automaton in Figure 1.2. Using this interpretation of L_2 and the set of generators it provided, they determined the spectrum of random walks on the binary tree, on which L_2 naturally acts. In [7], Grigorchuk, Linnell, Schick, and Żuk use this to show the existence of a closed Riemannian manifold with a rational L^2-Betti number, providing a counterexample to the strong form of Atiyah's conjecture [1]. Another common perspective comes from those interested in Diestel-Leader graphs—see [5, 14] for references therein—and the generalization to *cross-wired lamplighter groups*, in which changing the setting of one lamp may impact others [12, 19]. A broader introduction to lamplighter groups with further references to the literature can be found in Chapter 15 of [14].

The traditional lamplighter L_2 is an example of what is known as a self-similar group—see Section 3.3 and [13]. Self similar groups are groups generated by the states of a particular class of automata known as *invertible synchronous transducers*. We define these automata in Section 2.2, and they are also defined in [13, 17]. An automaton of this class is intuitively thought of as a machine with several states of operation. It receives streams of input and uses these to determine its output stream and its new state of operation. A great variety of groups are self similar—all finite groups, for a start—but the definition is not so broad as to be vacuous and several self-similar groups with interesting properties are easily described by their automaton. For instance, it was shown in [9] that the group generated by Figure 1.1 has growth rate between polynomial and exponential.

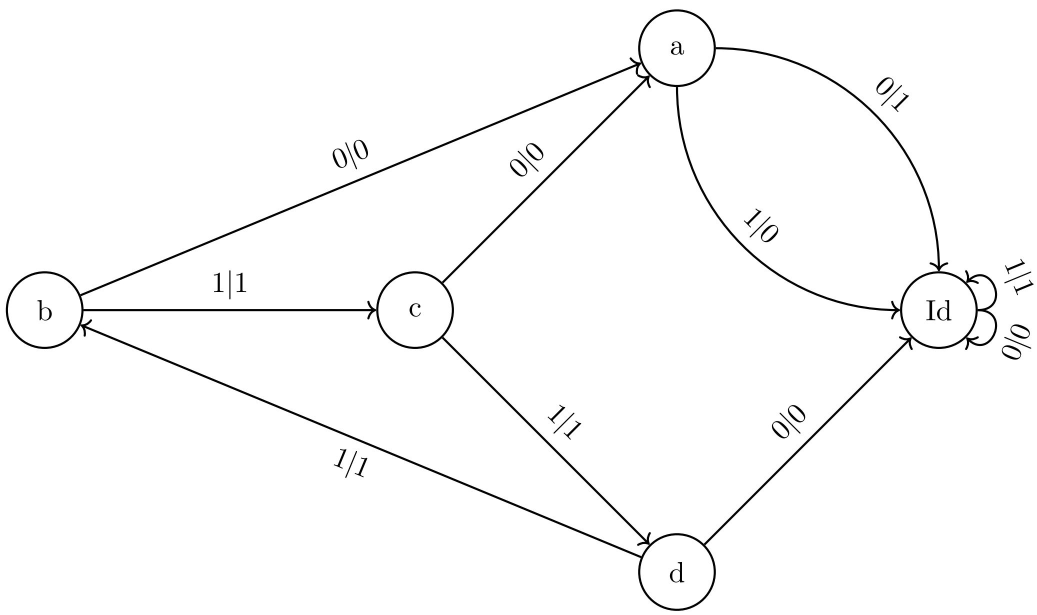

Figure 1.1: Automaton generating the Grigorchuk group

The automaton generating the lamplighter group L_2 is given by Figure 1.2. One might naturally wonder if the aforementioned generalizations of L_2 may be realized as groups generated by automata. This is the case for the first two generalizations, but not for the third—see Section 3.4, as well as [17].

The Cayley graph of the lamplighter group is a special case of a Diestel-Leader graph, which are constructed by "tying together" two regular trees oriented in opposing directions. The details of this are given in Section 3.1, as well as in [4] and [18].

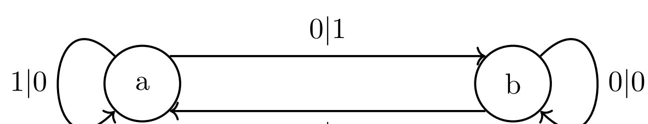

Figure 1.2: Automaton generating the lamplighter group

1.1 Notation

The conjugations $g^{-1}hg$ and ghg^{-1} are denoted (h^g) and $({}^g h)$, respectively, and the commutator $[g, h]$ is to mean $g^{-1}h^{-1}gh$. The collection of natural numbers is considered to be $\mathbb{N} = \{0, 1, \dots\}$, and $\mathbb{N}_+ = \{1, 2, \dots\}$. The restriction of a function $f : X \to Y$ to some set A is denoted $f|A$. In particular, A is not subscripted so as to avoid conflict with other notation introduced later in the paper. The identity function on a set A is denoted Id_A, and the symmetric group on A is denoted S_A.

Chapter 2: Algebraic Constructions

In what follows, three different constructions are discussed in service of the description of lamplighter groups—automata, tree automorphisms, and wreath products. The automata and the tree automorphisms are quite closely connected. In fact, the semigroups formed from tree endomorphisms are the semigroups generated by automata. Wreath products are a more general construction, but are well suited to represent the elements of self-similar groups and are used to define the lamplighter and diagonal groups. One might also refer to [13] for a different overview of all three constructions. Wreath products in particular are commonplace in graduate texts on group theory, see for example [15] and [16].

2.1 Wreath Products

Semi-direct products are used to build wreath products. As there are left and right semi-direct products, there are also left and right wreath products. There are times at which one is more natural than the other, and both find use in this paper.

Definition 1. Let G and H be groups and let $\varphi : H \to \operatorname{Aut} G$ be a homomorphism, with $\varphi(h)$ denoted φ_h. The *right semidirect product* $G \rtimes_\varphi H$ is the collection of pairs $(g, h) \in G \times H$ endowed with the multiplication $(g_1, h_1) \cdot (g_2, h_2) = (g_1\varphi_{h_1}(g_2), h_1h_2)$.

It is worth comparing semidirect and direct products:

1. If φ is the trivial homomorphism, then $G \times H \cong G \rtimes_\varphi H$.

2. In $G \times H$, the subgroups $G \cong G \times \{1\}$ and $H \cong \{1\} \times H$ are normal. Only G is normal in the semidirect product $G \rtimes_\varphi H$, unless φ is trivial.

3. In $G \times H$, the subgroup $[G, H] = \langle [g, h] : g \in G, h \in H\rangle$ is trivial, so G commutes with H in $G \times H$. This is not the case in $G \rtimes_\varphi H$ unless φ is trivial.

4. $G \cap H$ is trivial in $G \times H$ and in $G \rtimes_\varphi H$.

In fact, Condition 2 and Condition 4 provide the necessary and sufficient conditions for a group to be a semi-direct product.

Lemma 1. Let K be a group with subgroups $G, H \leq K$ such that $GH = K$. If G is normal in K and $G \cap H = \{1\}$, then there is a homomorphism $\varphi : H \to \operatorname{Aut} G$ for which $K \cong G \rtimes_\varphi H$.

Proof. As $K = GH$, that the map $K \to G \times H : gh \mapsto (g, h)$ is a bijection follows as soon as we show that it is well defined. But this is because each $k \in K$ has a unique representation as a product in GH—if the elements $g_1, g_2 \in G$ and $h_1, h_2 \in H$ are such that $g_1h_1 = g_2h_2 = k$, then $g_1g_2^{-1} = h_2h_1^{-1} = 1$ as $G \cap H = \{1\}$.

For every $h \in H$, let $\varphi : H \to \operatorname{Aut} G$ be given by $\varphi_h : g \mapsto ({}^h g) = hgh^{-1}$. Then given $k_1, k_2 \in K$ by $k_1 = g_1h_1$ and $k_2 = g_2h_2$ for appropriate $g_1, g_2 \in G$ and $h_1, h_2 \in H$, it follows

$$k_1k_2 = g_1h_1g_2h_2 = g_1h_1g_2h_1^{-1}h_1h_2 = g_1({}^{h_1}g_2)h_1h_2 = g_1\varphi_{h_1}(g_2)h_1h_2.$$

Thus $G \times H$ endowed with the multiplicative structure of K is $G \rtimes_\varphi H$. ■

Left semidirect products are defined using antihomomorphisms.

Definition 2. Let G, H be groups and let $\varphi : G \to H$ be such that $\varphi(g_1g_2) = \varphi(g_2)\varphi(g_1)$. Then φ is an *antihomomorphism*. These are closely related to right actions—if G right-acts on a set A, then the action induces an antihomomorphism from G to S_A, the symmetric group on A.

Definition 3. Let G and H be groups and let $\varphi : H \to \operatorname{Aut} G$ be an antihomomorphism, with $\varphi(h)$ denoted φ_h. Then the *left semidirect product* $H \ltimes_\varphi G$ is the collection of pairs $(h, g) \in H \times G$ endowed with the multiplication $(h_1, g_1) \cdot (h_2, g_2) = (h_1h_2, \varphi_{h_2}(g_1)g_2)$

Suppose again that K is a group with subgroups $G, H \leq K$ such that $HG = K$. If G is normal in K and $G \cap H = \{1\}$, there is an antihomomorphism $\varphi : H \to \operatorname{Aut} G$ for which

$K = H \ltimes_\varphi G$. In fact, it is $\varphi_h : g \mapsto g^h = h^{-1}gh$, as if k_1, k_2 are in K by $k_1 = h_1 g_1, k_2 = h_2 g_2$ for appropriate $g_1, g_2 \in G$ and $h_1, h_2 \in H$, then

$$k_1 k_2 = h_1 g_1 h_2 g_2 = h_1 h_2 h_2^{-1} g_1 h_2 g_2 = h_1 h_2 (g_1^{h_2}) g_2 = h_1 h_2 \varphi_{h_2}(g_1) g_2.$$

The rest of the proof is analogous to the one given for right wreath products.

Definition 4. If G and A are sets, then G^A denotes the collection of functions from A to G, equivalently the collection of A-tuples with entries from G. If G is a group, then this collection is referred to as the *direct power* of G by A. It is a group by the operation $(g_\alpha)_{\alpha \in A} \cdot (h_\alpha)_{\alpha \in A} = (g_\alpha h_\alpha)_{\alpha \in A}$ for each $(g_\alpha)_{\alpha \in A}, (h_\alpha)_{\alpha \in A} \in G^A$. The *direct sum* $\oplus_A G$ is the subgroup of G^A consisting of the elements $(g_\alpha)_{\alpha \in A} \in G^A$ in which $g_\alpha \neq 1$ holds for only finitely many α.

If A is finite, then the direct sum is equal to the direct power. For brevity, $(g_\alpha)_{\alpha \in A}$ may be shortened to $(g_\alpha)_\alpha$. If an element $(g_\alpha)_\alpha \in G^A$ is constantly g, we denote it $\vec{g}$.

Lemma 2. Let G and H be groups, and let A be a set. If $\varphi : H \to S_A$ is a homomorphism, then the induced function $H \to \operatorname{Aut} G^A$ given by $\varphi_h : (g_\alpha)_{\alpha \in A} \mapsto (g_{\varphi_h(\alpha)})_{\alpha \in A}$ is an antihomomorphism.

Proof. Given $(g_\alpha)_\alpha \in \operatorname{Aut} G^A$ and $h_1, h_2 \in H$, one computes

$$(\varphi_{(h_1 h_2)})((g_\alpha)_\alpha) = (g_{(\varphi_{(h_1 h_2)})(\alpha)})_\alpha = (g_{\varphi_{h_1}(\varphi_{h_2}(\alpha))})_\alpha = \varphi_{h_2}(g_{\varphi_{h_1}(\alpha)})_\alpha = \varphi_{h_2}(\varphi_{h_1}(g_\alpha)_\alpha). \quad \blacksquare$$

Corollary 1. If we restrict the domain of the induced function to $\oplus_A G$, then the induced function is an antihomomorphism into $\oplus_A G$.

Proof. As φ_h is in $\operatorname{Aut} G^A$, its effect on an element of G^A is to permute its entries. It follows that if $(g_\alpha)_\alpha$ is in $\oplus_A G$, then so too is the map $\varphi_h((g_\alpha)_\alpha) = (\varphi_h(g_\alpha))_\alpha$. $\blacksquare$

Similarly, if φ were an antihomomorphism, then the induced function would be a homomorphism. This corollary gives the final piece of information needed to define wreath products:

Definition 5. Let G and H be groups, and let A be a set. Let H right-act on A, inducing an antihomomorphism $\varphi : H \to S_A$ and thus a homomorphism $\varphi : H \to \operatorname{Aut} \oplus_A G$. The *right wreath product* $G \operatorname{wr} H$ is the right semidirect product $(\oplus_A G) \rtimes_\varphi H$.

If H instead left-acts on A, inducing a homomorphism $\varphi : H \to S_A$ and thus an antihomomorphism $\varphi : H \to \operatorname{Aut} G^A$, then the *left wreath product* $H \wr G$ is the left semidirect product $H \ltimes_\varphi (\oplus_A G)$.

Remark 1. Definition 5 provides what are known as *restricted* wreath products. *Unrestricted* wreath products are of form $G^A \rtimes_\varphi H$ (right) and $H \ltimes_\varphi G^A$ (left). Unrestricted wreath products are not used in this paper.

2.2 Automata

The second construction of note is that of automata. We are not interested in the study of these structures themselves, although they are quite interesting. Instead, the aim is to develop exactly enough of their theory to see how they can be used to generate groups.

Definition 6. An *automaton* is a quintuple $\mathcal{A} = (I, O, Q, \delta, \lambda)$. The *input alphabet* I and the *output alphabet* O are finite sets of elements known as *letters*. The set Q is of elements known as *states*. It is not generally true that Q is finite, but when it is then $\mathcal{A}$ is said to be *finite state*. The function $\delta : Q \times I \to Q$ is known as the *transition function*, and $\lambda : Q \times I \to O$ is known as the *output function*.

Example 1. A coffee machine can be modeled as the automaton $\mathscr{C}$, with

- The input alphabet $I = \{\text{grounds}, \text{water}, \text{power}\}$.
- The output alphabet $O = \{\text{nothing}, \text{grounds}, \text{water}, \text{coffee}, \text{fire}\}$—a fire is perhaps a somewhat dramatic output for a coffee machine run without any water, but it provides more states, and unattended coffee machines do catch fire on occasion.
- The states $Q = \{\text{empty}, \text{with grounds}, \text{with water}, \text{with both}, \text{broken}\}$.

The state and transition functions would be cumbersome to write out, so instead one uses a *Moore diagram*. Figure 2.1 provides the appropriate digraph. Here, the states are represented as circles, and, for a given state and input, the resulting state—the output from the transition function—is pointed to by an arrow from the initial state. The arrow is labeled with the input letter and output letter (from the output function), which are separated by a vertical bar.

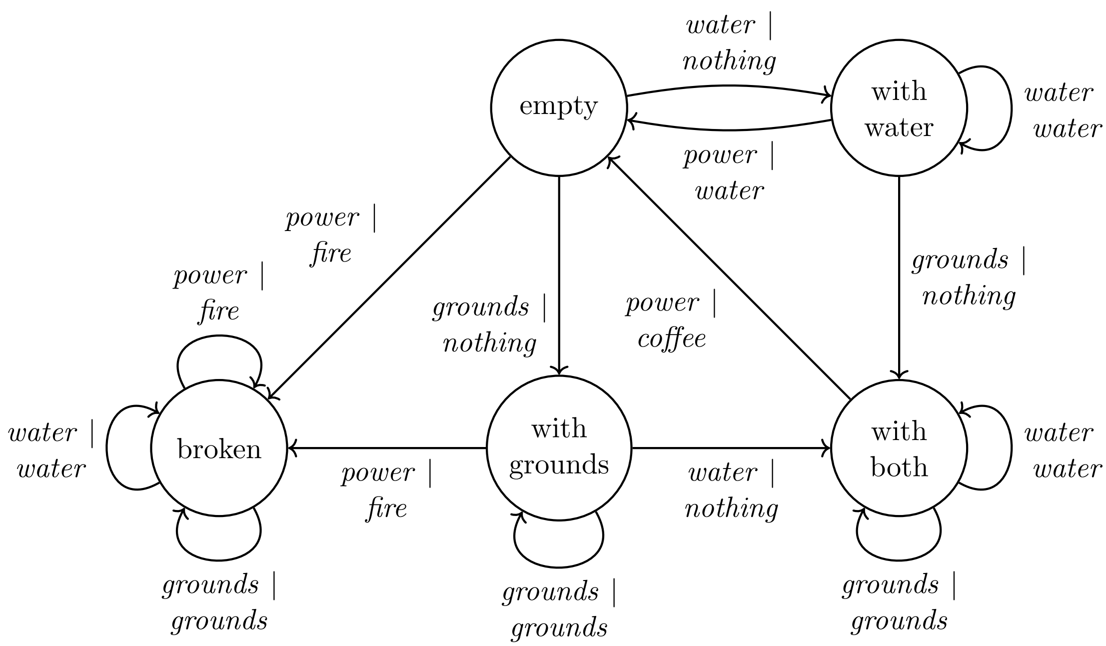

Figure 2.1: A coffee machine modeled as an automaton and described via a Moore diagram

Except in some examples, the input and output alphabets will be identical. When this is the case then $\mathcal{A}$ will be denoted as a quadruple and the common alphabet will be denoted A.

Every state $q \in Q$ is associated via the output function to a map $I \to O$ given by $\lambda_q(\,\cdot\,) = \lambda(q,\,\cdot\,)$ and via the transition function to a map $I \to Q$ given by $\delta(q,\,\cdot\,)$. We can then chain these together to associate to each $q \in Q$ functions $I^2 \to O^2$ and $I^2 \to Q$ as follows (with input $(\alpha, \beta) \in I^2$):

1. $q(\,\cdot\,) : (\alpha, \beta) \mapsto (\lambda(q, \alpha), \lambda(\delta(q, \alpha), \beta))$

2. $q|_{(\,\cdot\,)} : (\alpha, \beta) \mapsto \delta(\delta(q, \alpha), \beta)$

These maps correspond to taking two steps in the Moore diagram of the automaton. The first map provides the output element from each edge along the walk, and the second map provides the state landed on at the end of the walk.

Example 2. Returning to the context of our coffee machine $\mathscr{C}$, we have

1. $\lambda(\text{empty}, \textit{grounds}) = \textit{nothing}$,
2. $\delta(\text{empty}, \textit{grounds}) =$ "with grounds",
3. $\lambda(\text{with grounds}, \textit{power}) = \textit{fire}$, and
4. $\delta(\text{with grounds}, \textit{power}) =$ "broken".

With $q =$ "empty", $\alpha = \textit{grounds}$, and $\beta = \textit{power}$, the output under $q(\,\cdot\,)$ would be $(\textit{nothing}, \textit{fire})$, and the output under $q|_{(\,\cdot\,)}$ would be "broken".

Continuing in this manner gives, for each state $q \in Q$, an extension of $q(\,\cdot\,)$ (which we denote $q(\,\cdot\,)$ as well) from I^n to O^n. Moreover, if this extension sends $(\alpha_1, \ldots, \alpha_n)$ to $(\beta_1, \ldots, \beta_n)$ and sends $(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ to $(\gamma_1, \ldots, \gamma_n, \gamma_{n+1})$, then $\beta_i = \gamma_i$ for each i by induction. In this way, $q(\,\cdot\,)$ builds on its mapping of each $(\alpha_1, \ldots, \alpha_n)$. Viewing each sequence as a set of pairs of indices and values, we can extend $q(\,\cdot\,)$ to infinite I-sequences by taking

$$(\alpha_n)_{n \in \mathbb{N}_+} = \bigcup_{n \in \mathbb{N}_+} \{(i, \alpha_i) : 1 \leq i \leq n\}.$$

We can now begin the process of codifying $q(\,\cdot\,)$, $q|_{(\,\cdot\,)}$, and their extensions.

Definition 7. Let $A^0 = \{\emptyset\}$, where $\emptyset$ denotes the empty sequence. For each positive natural number n, let $A^{<n} = \bigcup_{i=0}^{n-1} A^i$ be the collection of A-sequences with length less than n, and let $A^{\leq n} = A^n \cup A^{<n}$ be the collection of A-sequences with length at most n. Define $A^{<\omega} = \bigcup_{i \in \mathbb{N}} A^i$ to be the collection of all finite A-sequences, and define $A^{\leq\omega} = A^\omega \cup A^{<\omega}$ to be the collection of finite and countable A-sequences.

Note that $\emptyset$ is in $A^{<n}$ for each $n \geq 1$, is in $A^{\leq n}$ for each $n \geq 0$, is in $A^{<\omega}$, and is in $A^{\leq\omega}$. In the context of the alphabet A of an automaton, Greek script is used to denote elements of A, while Latin script and sequence notation are used to denote elements of A^n, $A^{<\omega}$, and $A^{\leq\omega}$. We make an exception to this Greek-Latin dichotomy in certain cases. For example, it may not make sense to use different scripts if $A = Q$ or when referring to specific letters and words of a given alphabet.

Definition 8. Let $a \in A^{\leq\omega}$. It is called a *word*. The length of a is its length as a sequence and is denoted $|a|$. If $b \in A^{<\omega}$, the *concatenation* of $a = (\alpha_1, \alpha_2, \dots)$ to $b = (\beta_1, \dots, \beta_{|b|})$ is the sequence $(\beta_1, \dots, \beta_{|b|}, \alpha_1, \alpha_2, \dots)$, denoted $b^\frown a$ or ba. The latter form is used to denote elements of $A^{<\omega}$ as repeated concatenations of elements of A. The collections $A^{<\omega}$ and $A^{\leq\omega}$ are partially ordered as follows: given $a, b \in A^{\leq\omega}$, we say $a \leq b$ if $b = a^\frown c$ for some $c \in A^{\leq\omega}$. If this is the case, then a is called a *prefix* or *initial segment* of b. Under this ordering, $A^{\leq\omega}$ is a *set theoretic tree*, meaning for each $a \in A^{\leq\omega}$, the set $\{b \in A^{\leq\omega} : b < a\}$ is well-ordered.

Definition 9. For $q \in Q$ and $\alpha_1, \dots, \alpha_n \in I$, we denote $q|_{\alpha_1} = \delta(q, \alpha_1)$ and recursively define $(q|_{\alpha_1 \cdots \alpha_{n-1}})|_{\alpha_n} = q|_{\alpha_1 \cdots \alpha_n}$. Extending these maps to I^ω is not consistently well defined, because the state of the automaton given an infinite input sequence may not stabilize. We define $q(\alpha_1) = \lambda(q, \alpha_1)$, recursively define

$$q(\alpha_1 \cdots \alpha_n) = q(\alpha_1 \cdots \alpha_{n-1})^\frown (q|_{\alpha_1 \cdots \alpha_{n-1}})(\alpha_n),$$

then take the union

$$\bigcup_{n=1}^{\infty} \{(i, (q(\alpha_1 \cdots \alpha_n))_i) : i \leq n\} = q(\alpha_1)^\frown q|_{\alpha_1}(\alpha_2)^\frown \cdots$$

($q(\alpha_1 \cdots \alpha_n)$ is a sequence, so it can be indexed) to mean $q(\alpha_1 \alpha_2 \cdots)$. These conveniently express the generalizations of Item 1 and Item 2.

Thus, if $\mathcal{A} = (A, Q, \delta, \lambda)$ is an automaton, each $q \in Q$ corresponds to a function $A^{\leq\omega} \to$

$A^{\leq\omega}$ by $q : a \mapsto q(a)$. Each state $q \in Q$ is identified with the induced function $A^{\leq\omega} \to A^{\leq\omega} : a \mapsto q(a)$ except in rare cases, where the decoupling will be noted.

Example 3. It is not generally the case that distinct states in Q correspond to separate functions. For instance, the automaton with Moore diagram given by Figure 2.2,

$$\left(\{0\}, \{p, q\}, \begin{Bmatrix} (p,0) & : & p \\ (q,0) & : & q \end{Bmatrix}, \begin{Bmatrix} (p,0) & : & 0 \\ (q,0) & : & 0 \end{Bmatrix} \right)$$

is such that p and q both correspond to the constant function $\mathrm{Id}_{\{0\}}$.

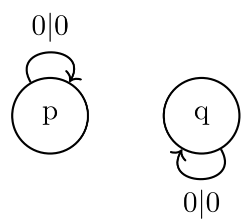

Figure 2.2: An automaton with two states corresponding to the same function

Definition 10. An automaton $\mathcal{A} = (I, O, Q, \delta, \lambda)$ is *reduced* if the correspondence between Q and functions $I^{\leq\omega} \to O^{\leq\omega}$ is injective.

Example 4. Figure 2.3 gives the automaton $\mathscr{M}$ of a different sort of coffee machine. This automaton is reduced, as the output on either element of the input alphabet $I = \{\textit{nothing}, \textit{coffee}\}$ differs between the two states $Q = \{\text{caffeinated}, \text{decaffeinated}\}$.

A particular class of automata are easily connected to groups:

Definition 11. Let $\mathcal{A} = (I, O, Q, \delta, \lambda)$ be an automaton. If $q : I \to O$ is a bijection for every state $q \in Q$, then we say $\mathcal{A}$ is *invertible*. The *inverse automaton* of $\mathcal{A}$ is defined to be $\mathcal{A}^{-1} = \left(O, I, Q^{-1}, \tilde{\delta}, \tilde{\lambda}\right)$. Here Q^{-1} is formally defined to be $\{q^{-1} : q \in Q\}$ and so is in bijective correspondence with Q. The transition and output functions of $\mathcal{A}^{-1}$ are such that

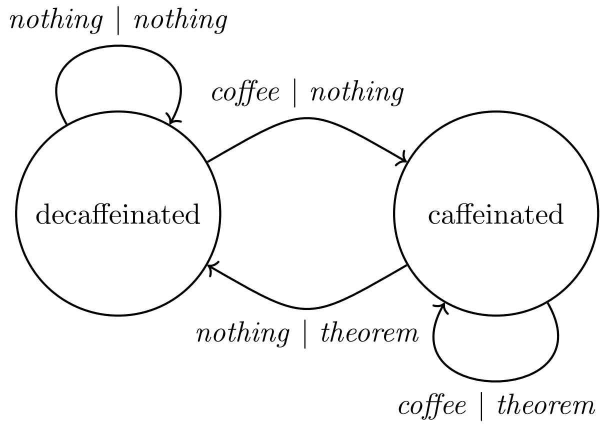

Figure 2.3: A different sort of coffee machine

$\tilde{\delta}(q^{-1}, \alpha) = p^{-1}$ and $\tilde{\lambda}(q^{-1}, \alpha) = \beta$ if and only if $\delta(q, \beta) = p$ and $\lambda(q, \beta) = \alpha$. That is to say, $q^{-1}(\alpha) = \beta$ and $q^{-1}|_{\alpha} = p^{-1}$ if and only if $q(\beta) = \alpha$ and $q|_{\beta} = p$.

Example 5. The automaton $\mathscr{M}$ of Figure 2.3 is not an invertible automaton, as both of its states are constant on I (though not on $I^{\leq\omega}$). The automaton $\mathscr{C}$ of Figure 2.1 is also not invertible as it has a larger output alphabet than input alphabet.

Example 6. Let G be a finite group, let $A = Q = G$, let $\lambda : G \to G$ be given by $\lambda(g, h) = gh$, and let $\delta : G \to G$ be given by $\delta(g, h) = 0$ for all $g, h \in G$. This automaton is invertible as $\lambda_g = \lambda(g, \cdot)$ is invertible for each $g \in G$. See Figure 2.4 for this automaton when $G = \mathbb{Z}/3\mathbb{Z}$. As a small remark, this is the automaton that generates G (although we have not yet defined what this means).

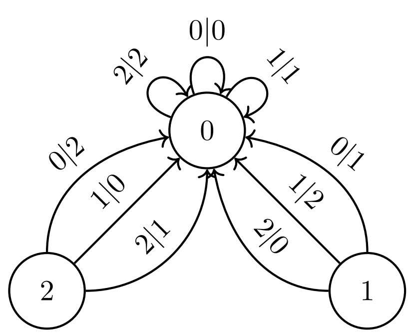

Figure 2.4: The automaton that generates $\mathbb{Z}/3\mathbb{Z}$, as described in Example 6

Example 7. Take the Cayley graph of a finite group G, generated by G itself, and for each pair of elements $g, h \in G$, label the arrow from g to h with input $g^{-1}h$ and output h. The result is an invertible automaton, denoted $\mathcal{C}(G)$. This construction is given in Figure 2.5 for the dihedral group $D_{2\cdot 3} = \langle r, s : r^3, s^2, (rs)^2 \rangle$. It is known as a *Cayley machine* [17]. Cayley machines are used extensively in Chapter 3.

Remark 2. If $\mathcal{A} = (A, Q, \delta, \lambda)$ is an invertible automaton, then $q^{-1}(q(\alpha)) = \alpha$ and $q^{-1}|_\alpha = q|_\alpha^{-1}$ for any letter $\alpha \in A$ and state $q \in Q$ by Definition 11. In particular, $(q^{-1} \circ q) = \mathrm{Id}_{A^{\leq\omega}}$.

Definition 12. A map $f : A^{\leq\omega} \to A^{\leq\omega}$ is called a (rooted-)*tree automorphism* if f is bijective, preserves the length of words, and preserves prefixes, in that if $a \in A^{<\omega}$ and $u \in A^{\leq\omega}$, then $f(a^\frown u) = f(a)^\frown v$ for some $v \in A^{\leq\omega}$. The group of tree automorphisms of $A^{\leq\omega}$ is denoted $\mathrm{Aut}\, A^{\leq\omega}$.

Definition 13. If $\mathcal{A} = (A, Q, \delta, \lambda)$ is an invertible automaton, then each state $q \in Q$ is in $\mathrm{Aut}\, A^{\leq\omega}$ by Definition 9 and Remark 2. The phrasing $Q \subseteq \mathrm{Aut}\, A^{\leq\omega}$ is avoided unless $\mathcal{A}$ is reduced, as otherwise multiple states of Q correspond to the same automorphism. If q is a state in Q (and so an automorphism of $A^{\leq\omega}$) and a is a word in $A^{<\omega}$, then $q|_a$ is necessarily a state in Q, so $q|_a$ is also in $\mathrm{Aut}\, A^{\leq\omega}$.

Definition 14. We extend the notation $q|_{(\cdot)}$ to tree automorphisms, so if $f \in \mathrm{Aut}\, A^{\leq\omega}$ is a tree automorphism and $a \in A^{<\omega}$ is a word, then $f|_a$ is such that $f(a^\frown b) = f(a)^\frown f|_a(b)$ for all words $b \in A^{\leq\omega}$. It is also true that $f|_a \in \mathrm{Aut}\, A^{\leq\omega}$, however if G is a subgroup of $\mathrm{Aut}\, A^{\leq\omega}$ and f is a tree automorphism in G, it is not necessarily true that $f|_a$ remains in G for all words $a \in A^{<\omega}$.

Example 8. Let $A = \mathbb{Z}/2\mathbb{Z}$ and let $f \in \mathrm{Aut}\, A^{\leq\omega}$ by

$$f(\alpha_1\alpha_2\alpha_3 \cdots \alpha_n \cdots) = \alpha_1^\frown(\alpha_2 + 1)^\frown(\alpha_3 + 1)^\frown \cdots ^\frown (\alpha_n + 1)^\frown \cdots$$

where the addition is modulo 2. Then $\{\mathrm{Id}_{A^{\leq\omega}}, f\} \leq \operatorname{Aut} A^{\leq\omega}$ as

$$\begin{aligned} f^2(\alpha_1\alpha_2\alpha_3\cdots\alpha_n\cdots) &= f(\alpha_1{}^\frown(\alpha_2+1)^\frown(\alpha_3+1)^\frown\cdots{}^\frown(\alpha_n+1)^\frown\cdots) \\ &= \alpha_1{}^\frown(\alpha_2+2)^\frown(\alpha_3+2)^\frown\cdots{}^\frown(\alpha_n+2)^\frown\cdots \\ &= \alpha_1{}^\frown\alpha_2{}^\frown\cdots{}^\frown\alpha_n{}^\frown\cdots \\ &= \mathrm{Id}_{A^{\leq\omega}}(\alpha_1\alpha_2\alpha_3\cdots\alpha_n\cdots). \end{aligned}$$

As $f|_0 = f|_1 = (\alpha_n)_n \mapsto (\alpha_n+1)_n$, neither $f|_0$ nor $f|_1$ is in $\{\mathrm{Id}_{A^{\leq\omega}}, f\}$.

Definition 13 and Example 8 point to what is notable about using an automaton to obtain a subset of $\operatorname{Aut} A^{\leq\omega}$. We give a name to groups and semigroups generated by the states of an automaton.

Definition 15. Let $\mathcal{A} = (A, Q, \delta, \lambda)$ be an automaton. Then the *semigroup generated by* $\mathcal{A}$, denoted $\mathsf{S}(\mathcal{A})$, is the smallest subsemigroup of $\operatorname{Aut} A^{\leq\omega}$ containing, for each state $q \in Q$, the map associated to q by Definition 9. If $\mathcal{A}$ is invertible, then the *group generated by* $\mathcal{A}$, denoted $\mathsf{G}(\mathcal{A})$, is the smallest subgroup of $\operatorname{Aut} A^{\leq\omega}$ containing the collection of maps associated with the collection of states Q. By Remark 2, $\mathsf{G}(\mathcal{A}) = \mathsf{G}(\mathcal{A}^{-1})$.

Such a group, or one isomorphic to it, is said to be *self-similar*. If $\mathcal{A}$ is finite-state, then the group is called an *automaton group*.

Moore diagrams allow us to easily see the input-output and input-state relationships of a given state. While one can define products of automata and interpret the products through Moore diagrams (see Section 1.3 of [13]), it is cumbersome to work with. However, the next theorem shows that we can use wreath products to write the elements of a self similar group in a way that provides this same information. This is referred to as *wreath recursion* in Section 1.4 of the same source.

Theorem 1. Let $\mathcal{A} = (A, Q, \delta, \lambda)$ be an invertible automaton. $\operatorname{Aut} A^{\leq\omega} \cong S_A \wr \operatorname{Aut} A^{\leq\omega}$, so $\mathsf{G}(\mathcal{A}) \leq S_A \wr \operatorname{Aut} A^{\leq\omega}$.

Proof. Two tree automorphisms $f, g \in \operatorname{Aut} A^{\leq\omega}$ differ if and only if $f|A \neq g|A$ or $f|_\alpha \neq g|_\alpha$ for some $\alpha \in A$. Thus it suffices to show that every $f \in \operatorname{Aut} A^{\leq\omega}$ is representable as $(f|A, (f|_\alpha)_{\alpha\in A})$ and that the multiplications agree. If $f, g \in \operatorname{Aut} A^{\leq\omega}$ correspond to the same representation, then $f(\alpha) = g(\alpha)$ for any $\alpha \in A$, and as $f(\alpha_1\alpha_2\cdots) = f(\alpha_1)^\frown f|_{\alpha_1}(\alpha_2\cdots) = g(\alpha_1)^\frown g|_{\alpha_1}(\alpha_2\cdots)$ it follows $f = g$. Then compute

$$\begin{aligned}(f \circ g)(\alpha_1\alpha_2\cdots) &= f(g(\alpha_1\alpha_2\cdots))\\ &= f(g(\alpha_1)^\frown g|_{\alpha_1}(\alpha_2\cdots))\\ &= f(g(\alpha_1))^\frown f|_{g(\alpha_1)}(g(\alpha_2\cdots))\\ &= ((f\circ g)|A)(\alpha_1)^\frown(f|_{g(\alpha_1)} \circ g|_{\alpha_1})(\alpha_2\cdots)\end{aligned}$$

so $(f\circ g)$ corresponds to $((f\circ g)|A, (f|_{g(\alpha)}\circ g|_\alpha)_\alpha)$, the product of the corresponding elements in the wreath product. ■

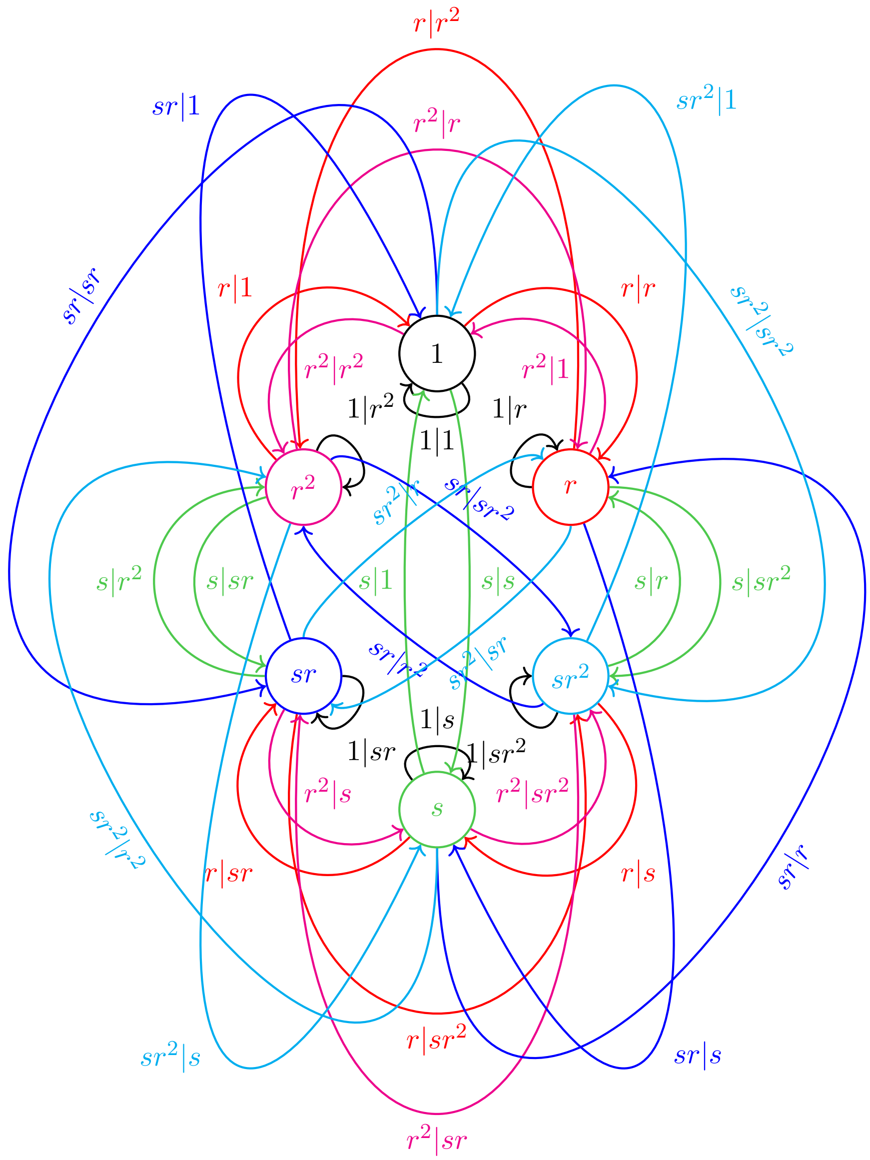

Figure 2.5: The Cayley machine of $D_{2\cdot 3}$

Chapter 3: Lamplighter Groups and Their Generalizations

Thus far we have defined wreath products, automata, and rooted tree automorphisms. We have seen how invertible automata can be used to generate groups, and how the elements of these groups can be represented as elements of a wreath product. This chapter begins by providing the definition for the collection of lamplighter groups. Following this, we establish presentations and Cayley graphs for the groups. It is interesting that the lamplighter is not finitely presentable—see [2]. We present an elaboration on the proof from [13] to show that the traditional lamplighter group L_2 is an automaton group. Following [17] and [10], we show exactly when this result can be generalized. In particular, the lamplighter group L_G associated with the finite group G is an automaton group if and only if G is Abelian. When G is Abelian, L_G is the group generated by the Cayley machine $\mathcal{C}(G)$ of G. It is then natural to ask what group $\mathcal{C}(G)$ generates when G is finite non-Abelian. In [17], Silva and Steinberg show that the resulting group is a semidirect product $N \rtimes \mathbb{Z}$, with N locally finite.

3.1 Presentations of Lamplighter Groups

Definition 16. Let G be a finite group. The *lamplighter group* associated with G is $L_G = G \operatorname{wr} \mathbb{Z} = (\oplus_{\mathbb{Z}} G) \rtimes_{\varphi} \mathbb{Z}$, with $\varphi : \mathbb{Z} \to \operatorname{Aut}(\oplus_{\mathbb{Z}} G)$ given by $\varphi(i) = ((g_n)_{n\in\mathbb{Z}} \mapsto (g_{n-i})_{n\in\mathbb{Z}})$.

The elements of L_G have the form $((g_n)_{n\in\mathbb{Z}}, i)$, with $g_n \neq 1_G$ for only finitely many $n \in \mathbb{Z}$. The multiplication is $((g_n)_{n\in\mathbb{Z}}, i)((h_n)_{n\in\mathbb{Z}}, j) = ((g_n h_{n-i})_{n\in\mathbb{Z}}, i + j)$. If $G = \mathbb{Z}/m\mathbb{Z}$ for some $m \in \mathbb{N}_+$, then L_G is denoted L_m.

For now, we stay in the context of L_2. For each $i \in \mathbb{Z}$, let $(\delta_{ij})_j \in \oplus_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}$ denote the list such that δ_{ij} equals 1 when $i = j$ and equals 0 otherwise—that is, δ_{ij} is the Kronecker delta. Let $s = ((\delta_{0j})_j, 0)$ and $w = (\vec{0}, 1) = ((0, 0, \dots), 1)$, where s stands for "switch" and w for "walk". We start with a lemma:

Lemma 3. Let $G = \langle u, v \rangle$, where $v^2 = 1$. Then every element $g \in G$ is such that $g = ({}^{u^{\sigma_1}}v) \cdots ({}^{u^{\sigma_n}}v)u^{\sigma_n} = (u^{\sigma_1} v u^{-\sigma_1}) \cdots (u^{\sigma_n} v u^{-\sigma_n})u^{\sigma_n}$ for appropriate $\sigma_1, \dots, \sigma_n \in \mathbb{Z}$.

Proof. As $G = \langle u, v \rangle$ and $v^2 = 1$, it follows that $g = u^{\ell_1} v \cdots u^{\ell_n} v$ for some $\ell_1, \dots, \ell_n \in \mathbb{Z}$ (if there is no trailing v then $\ell_n = 0$). If we let $\sigma_j = \sum_{i=1}^{j} \ell_i$ and insert $(u^{-\sigma_j})(u^{\sigma_j})$ after $u^{\ell_j} v$ for each $1 \le j \le n$, then

$$
\begin{aligned}
g &= u^{\ell_1} v \cdots u^{\ell_n} v \\
&= (u^{\ell_1} v)(u^{-\sigma_1})(u^{\sigma_1}) \cdots (u^{-\sigma_{n-1}})(u^{\sigma_{n-1}})(u^{\ell_n} v)(u^{-\sigma_n})(u^{\sigma_n}) \\
&= u^{\sigma_1} v u^{-\sigma_1} \cdots u^{\sigma_n} v u^{-\sigma_n}(u^{\sigma_n}) \\
&= ({}^{u^{\sigma_1}}v) \cdots ({}^{u^{\sigma_n}}v) u^{\sigma_n}.
\end{aligned}
$$

■

Theorem 2. The traditional lamplighter group L_2 is generated by the elements w and s under the relations $\{s^2\} \cup \{[{}^{(w^k)}s, {}^{(w^\ell)}s] : k, \ell \in \mathbb{Z}\}$, where ${}^{(w^k)}s = w^k s w^{-k}$.

Proof. First compute

$$
wsw^{-1} = (\vec{0}, 1)((\delta_{0j})_j, 0)(\vec{0}, -1) = ((\delta_{0,j-1})_j, 0) = ((\delta_{1j})_j, 0).
$$

By induction, every $(\delta_{ij})_j, 0)$ is in $\langle w, s \rangle$. For every finite list $i_1, \dots, i_k \in \mathbb{Z}$,

$$
((\delta_{i_1,j})_j, 0)((\delta_{i_k,j})_j, 0) = ((\delta_{i_1,j} + \cdots + \delta_{i_k,j})_j, 0)
$$

so $(\oplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \times \{0\} \le \langle w, s \rangle$. Right multiplying by $n = w^n$ then gives $\langle w, s \rangle = L_2$. As for relations, $s^2 = ((\delta_{0j})_j, 0)((\delta_{0j})_j, 0) = ((\delta_{0j} + \delta_{0j})_j, 0) = (\vec{0}, 0)$, and for any $k, \ell \in \mathbb{Z}$,

$$
\begin{aligned}
{}^{(w^k)}s \cdot {}^{(w^\ell)}s &= ((\delta_{kj})_j, 0)((\delta_{\ell j})_j, 0) \\
&= ((\delta_{kj} + \delta_{\ell j})_j, 0) \\
&= ((\delta_{\ell j} + \delta_{kj})_j, 0) \\
&= ((\delta_{\ell j})_j, 0)((\delta_{kj})_j, 0) \\
&= ({}^{w^\ell}s)({}^{w^k}s).
\end{aligned}
$$

Thus L_2 has the infinite collection of relations s^2 and $\left[{}^{(w^k)}s, {}^{(w^\ell)}s\right]$, where $k, \ell \in \mathbb{Z}$. It remains to show that this collection is sufficient. Let ${}^{(w^{\sigma_1})}s \cdots {}^{(w^{\sigma_\ell})}s \cdot w^{\sigma_\ell} \in L_2$—every element of L_2 has this form by Lemma 3—and suppose that the product vanishes. Then

$$
(\vec{0}, 0) = {}^{(w^{\sigma_1})}s \cdots {}^{(w^{\sigma_\ell})}s \cdot w^{\sigma_\ell} = ((\delta_{\sigma_1, j})_j, 0) \cdots ((\delta_{\sigma_\ell, j})_j, 0) \cdot (\vec{0}, \sigma_\ell) = \left(\left(\sum_{i=1}^{\ell} \delta_{\sigma_i, j} \right)_j, \sigma_\ell \right).
$$

Thus $\sigma_\ell = 0$, so ${}^{(w^{\sigma_1})}s \cdots {}^{(w^{\sigma_\ell})}s \cdot w^{\sigma_\ell} = {}^{(w^{\sigma_1})}s \cdots {}^{(w^{\sigma_\ell})}s$. For every $1 \leq q \leq \ell$, there must be some $r \neq q$ for which $\sigma_q = \sigma_r$, or else the sum $(\sum_{i=1}^{\ell} \delta_{\sigma_i, j})_j$ is nonvanishing. In fact, we can place the σ_i's into pairs $(\sigma_{q_1}, \sigma_{r_1}), \ldots, (\sigma_{q_m}, \sigma_{r_m})$ with $\sigma_{q_i} = \sigma_{r_i}$ for all $1 \leq i \leq m$. By repeated application of the established commutator relations:

$$
{}^{(w^{\sigma_1})}s \cdots {}^{(w^{\sigma_\ell})}s = \left({}^{(w^{\sigma_{q_1}})}s \cdot {}^{(w^{\sigma_{r_1}})}s \right) \cdots \left({}^{(w^{\sigma_{q_m}})}s \cdot {}^{(w^{\sigma_{r_m}})}s \right) = 1. \qquad \blacksquare
$$

More generally, one has

Theorem 3. Let $G = \langle S : R \rangle$ be a finite group. Then $L_G = \left\langle w, \tilde{S} \right\rangle$ with relations

$$
\tilde{R} \cup \left\{ \left[{}^{(w^k)}\tilde{s}, {}^{(w^\ell)}\tilde{t} \right] : k, \ell \in \mathbb{Z}, k \neq \ell, \tilde{s}, \tilde{t} \in \tilde{S} \right\},
$$

where $\tilde{S} = \{((s^{\delta_{0j}})_j, 0) : s \in S\}$ and $\tilde{R} = \{\tilde{s}_1^{k_1} \cdots \tilde{s}_\ell^{k_\ell} : s_1^{k_1} \cdots s_\ell^{k_\ell} \in R\}$.

Proof. The simplification of ${}^{w^{\sigma_1}}\tilde{s}_1 \cdots {}^{w^{\sigma_\ell}}\tilde{s}_\ell$ must be modified somewhat from what was done in Theorem 2. Instead of using the commutator relations to pair off σ_i's, use them to collect all ${}^{\sigma_j}\tilde{s}_i$'s with equal conjugating element together. Each collection of conjugations with equal exponent can be rewritten as a single conjugation of a string of $\tilde{s}_i$'s. If the product vanishes, then this string must belong to $\tilde{R}$. ■

3.2 Cayley Graphs of Lamplighter Groups

For a fixed positive integer q and a group G of order q, there is a node and edge labeling of the *Diestel Leader graph* $\mathrm{DL}(q, q)$ which makes it a Cayley graph of L_G under a certain set of generators. These Diestel Leader graphs arise by tying together two q-regular trees in a particular fashion. See [4, 18], both of which heavily influence this section.

Definition 17. Let $\mathbb{T}_q$ denote the $(q+1)$-regular tree. We identify $\mathbb{T}_q$ with its vertex set. The unique path connecting $x, y \in \mathbb{T}_q$ is called a *geodesic segment* and is denoted $\overline{xy}$. The tree $\mathbb{T}_q$ is metrized by the graph distance d, which in this case is given by the length of the segment between two nodes. That is, $d(x, y) = \mathrm{length}(\overline{xy})$.

Example 9. Figure 3.1 shows a geodesic segment $\overline{xy}$ in a 3-regular tree by highlighting the edges of the segment red. In the example, $d(x, y) = 4$.

Definition 18. A *geodesic ray* is an infinite path in $\mathbb{T}_q$. Two rays are *equivalent* if the symmetric difference between their ranges is finite, and a class under this relation is called an *end*. The representative of the end ε starting at $x \in \mathbb{T}_q$ is denoted $\overline{x\varepsilon}$.

Example 10. Figure 3.2 shows the beginning of the two geodesic rays $\overline{x\varepsilon}$ and $\overline{y\varepsilon}$ in the 3-regular tree. The edges belonging solely to $\overline{x\varepsilon}$, $\overline{y\varepsilon}$, and both are highlighted red, blue, and purple (respectively).

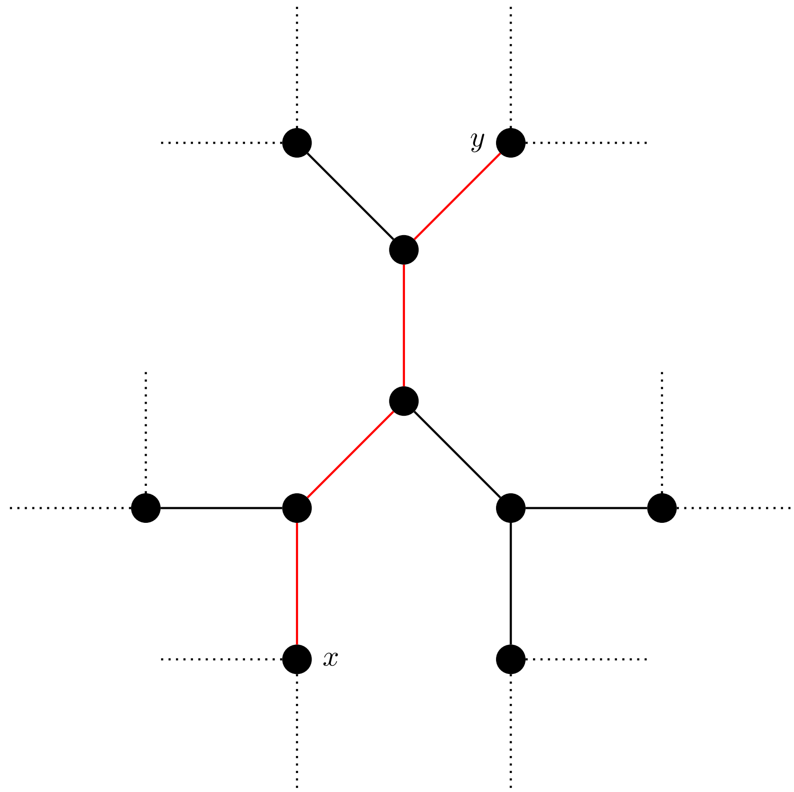

Figure 3.1: The geodesic segment between the nodes x and y from Example 9

Definition 19. The set of ends of $\mathbb{T}_q$ is denoted $\partial\mathbb{T}_q$, and we define $\hat{\mathbb{T}}_q = \mathbb{T}_q \cup \partial\mathbb{T}_q$.

Definition 20. Given a node $x \in \mathbb{T}_q$ and a node or end $\varepsilon \in \hat{\mathbb{T}}_q$, the i^{th} vertex of the path $\overline{x\varepsilon}$ is denoted $x_\varepsilon(i)$. Here, $x_\varepsilon(0) = x$.

Definition 21. Fix $\mathfrak{o} \in \mathbb{T}_q$ and $-\infty \in \partial\mathbb{T}_q$. The former is referred to as the *origin* or *root* of $\mathbb{T}_q$. Let $\zeta, \varepsilon, \delta, \gamma \in \hat{\mathbb{T}}_q$—in particular, it does not matter if they are nodes or ends. We say γ is the ζ-*confluent* of ε and δ if the largest common affix of of $\overline{\varepsilon\zeta}$ and $\overline{\delta\zeta}$ is $\overline{\gamma\zeta}$. The *origin-confluent* of ε and δ is denoted $\varepsilon \wedge \delta$, and the *end-confluent* ($-\infty$-confluent) is denoted $\varepsilon \curlywedge \delta$.

There is nothing special about the end $-\infty$ compared to any other, but later in this section we will want to orient two trees in opposing directions, so labeling the ends with the two signed infinities makes a certain sort of sense.

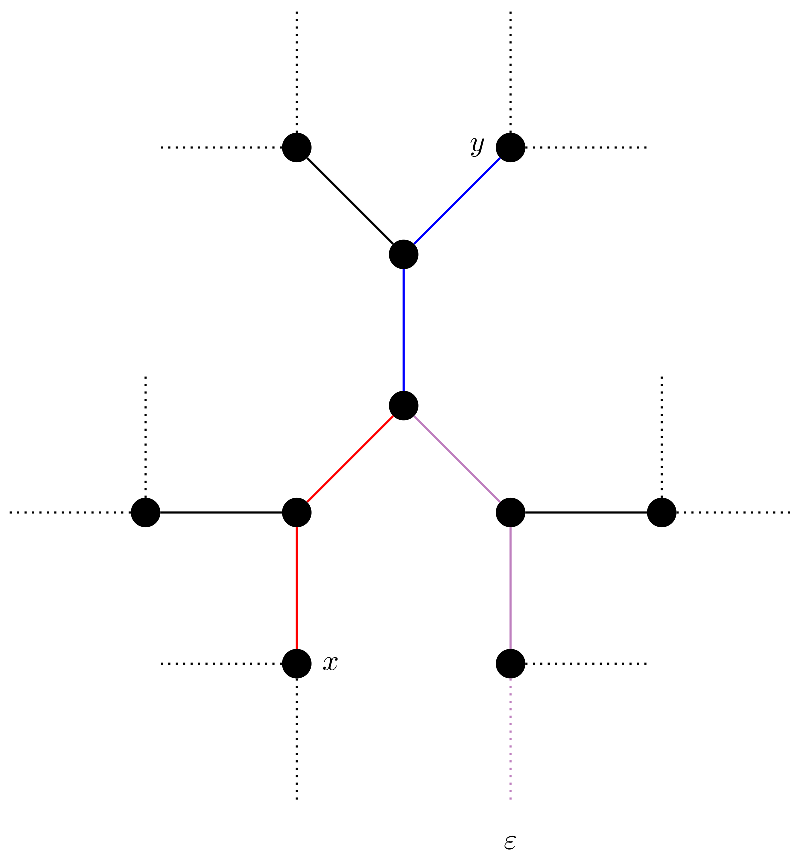

Figure 3.2: The geodesic rays $\overline{x\varepsilon}$ and $\overline{y\varepsilon}$ from Example 10

Example 11. Figure 3.3 shows a tree with labeled ends $\varepsilon, -\infty$ and labeled nodes $\mathfrak{o}$, x, y, $\varepsilon \wedge x$, $\mathfrak{o} \curlywedge y$. The pairs of paths $\overline{\mathfrak{o}x}$, $\overline{\mathfrak{o}\varepsilon}$ and $\overline{\mathfrak{o}(-\infty)}$, $\overline{y(-\infty)}$ are highlighted with contrasting colors so their overlap is grey.

Remark 3. The ζ-confluent of ε and δ is an end if and only if $\varepsilon = \delta$ and ε, δ are ends themselves.

Definition 22. Choose a second end $\varepsilon \in \partial\mathbb{T}$ which is such that $\overline{\mathfrak{o}(-\infty)}$ and $\overline{\mathfrak{o}\infty}$ do not meet except at $\mathfrak{o}$. Thus $\overline{(-\infty)\varepsilon}$ is a bi-infinite path containing $\mathfrak{o}$. Label every node along this path with 0, and for each now-labeled-node, label its unlabeled neighbors with indices $1, \ldots, q-1$. Then recursively, for every node with unlabeled neighbors, label said neighbors with $0, \ldots, q-1$. With our example 3-regular tree, this might look as in Figure 3.4. To

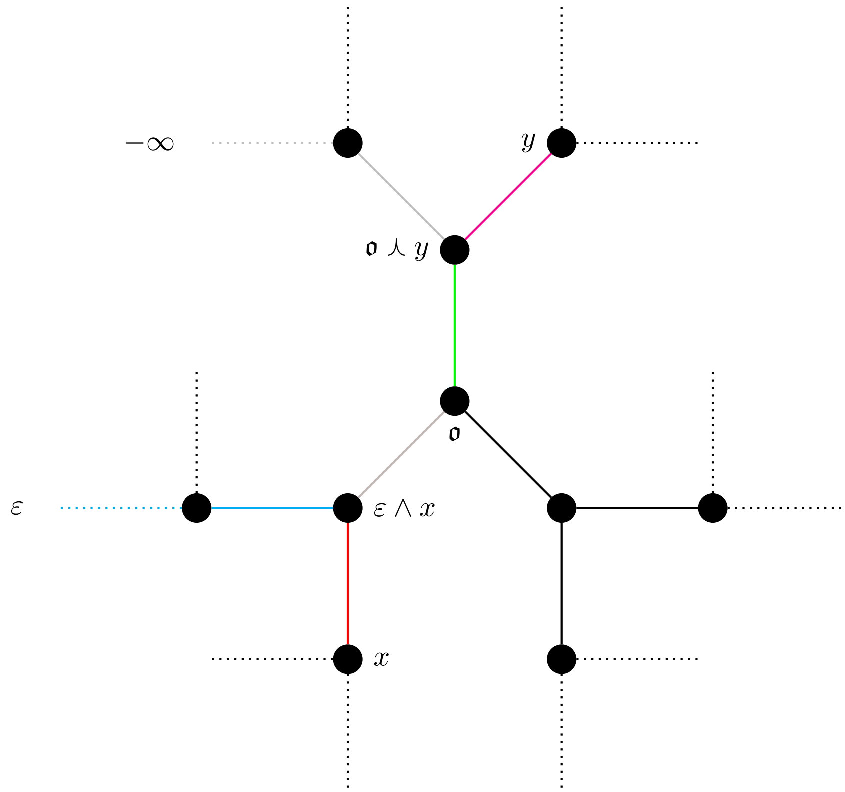

Figure 3.3: The confluents $\varepsilon \wedge x$ and $\mathfrak{o} \curlywedge y$ from Example 11

make the process clearer, the nodes are colored, with the colors moving from the inner edge to the outer edge of the rainbow as the iteration on which they were labeled increases.

Definition 23. The $(-\infty$-$)$*Busemann function* $\mathfrak{h} : \mathbb{T}_q \to \mathbb{Z}$ is defined as

$$\mathfrak{h}(x) = d(x, x \curlywedge \mathfrak{o}) - d(\mathfrak{o}, x \curlywedge \mathfrak{o}).$$

The collections $H_n = \mathfrak{h}^{-1}[\{n\}]$, where n is an integer, are known as $(-\infty$-$)$*horocycles*. Figure 3.5 reorganizes the 3-regular tree as labeled in Figure 3.4 so that the elements of a given horocycle lie upon the same horizontal. Note that $\mathfrak{h}(\mathfrak{o}) = 0$.

Lemma 4. For every $x \in \mathbb{T}_q$ and $i \in \mathbb{N}$, the Busemann function $\mathfrak{h}(x_{-\infty}(i))$ of the i^{th} entry along the path $\overline{x(-\infty)}$ is given by $\mathfrak{h}(x) - i$.

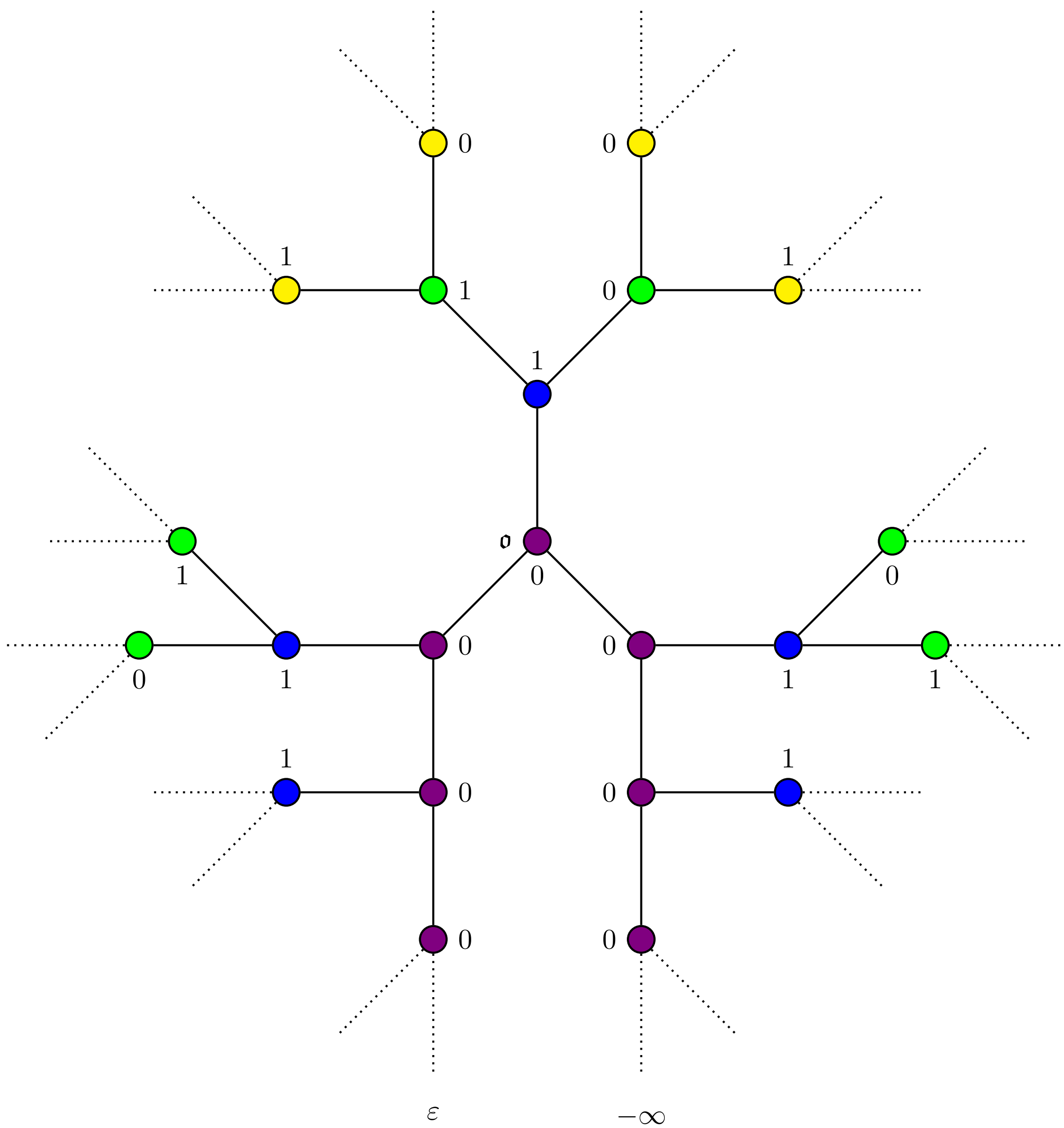

Figure 3.4: The 3-regular tree, labeled as described in Definition 22

Proof. Let $c = x \curlywedge \mathfrak{o}$. If $x_{-\infty}(i)$ is such that $\overline{x(-\infty)} = \overline{x(x_{-\infty}(i))}^\frown\overline{(x_{-\infty}(i))c}^\frown\overline{c-\infty}$, then $x_{-\infty}(i)$ is along the path $\overline{xc}$ and—as otherwise $\overline{c(-\infty)}$ would not be the largest common affix of $\overline{x(-\infty)}$ and $\overline{\mathfrak{o}(-\infty)}$, it follows $x_{-\infty}(i) \curlywedge \mathfrak{o} = c$. Then

$$\begin{aligned} d(x_{-\infty}(i), x_{-\infty}(i) \curlywedge \mathfrak{o}) &= d(x_{-\infty}(i), c) \\ &= \text{length}(\overline{xc}) - \text{length}\left(\overline{x(x_{-\infty}(i))}\right) \\ &= d(x, c) - i \end{aligned}$$

and $d(\mathfrak{o}, x_{-\infty}(i) \curlywedge \mathfrak{o}) = d(\mathfrak{o}, x \curlywedge \mathfrak{o})$, giving $\mathfrak{h}(x_{-\infty}(i)) = \mathfrak{h}(x) - i$.

If instead $x_{-\infty}(i)$ is such that $\overline{x(-\infty)} = \overline{xc}^\frown\overline{c(x_{-\infty}(i))}^\frown\overline{(x_{-\infty}(i))(-\infty)}$, then the path $\overline{(x_{-\infty}(i))(-\infty)}$, being an affix of $\overline{c(-\infty)}$, is the largest common affix of $\overline{(x_{-\infty}(i))(-\infty)}$ and $\overline{\mathfrak{o}(-\infty)}$. Thus $d(x_{-\infty}(i), x_{-\infty}(i) \curlywedge \mathfrak{o}) = d(x_{-\infty}(i), x_{-\infty}(i)) = 0$ and

$$\begin{aligned} d(\mathfrak{o}, x_{-\infty}(i) \curlywedge \mathfrak{o}) &= d(\mathfrak{o}, x_{-\infty}(i)) \\ &= \text{length}(\overline{\mathfrak{o}c}) + \text{length}\left(\overline{c(x_{-\infty}(i))}\right) + (\text{length}(\overline{xc}) - \text{length}(\overline{xc})) \\ &= \text{length}(\overline{\mathfrak{o}c}) - \text{length}(\overline{xc}) + \left(\text{length}(\overline{xc}) + \text{length}\left(\overline{c(x_{-\infty}(i))}\right)\right) \\ &= d(\mathfrak{o}, x \curlywedge \mathfrak{o}) - d(x, x \curlywedge \mathfrak{o}) + i \\ &= -\mathfrak{h}(x) + i, \end{aligned}$$

which gives that

$$\begin{aligned} \mathfrak{h}((x_{-\infty}(i))) &= d((x_{-\infty})(i), (x_{-\infty})(i) \curlywedge \mathfrak{o}) - d(\mathfrak{o}, (x_{-\infty})(i) \curlywedge \mathfrak{o}) \\ &= 0 - (-\mathfrak{h}(x) + i) \\ &= \mathfrak{h}(x) + i \end{aligned}$$

■

Definition 24. Given a node x of $\mathbb{T}_q$ in the horocycle $\mathfrak{h}(x)$, we refer to the q neighbors y

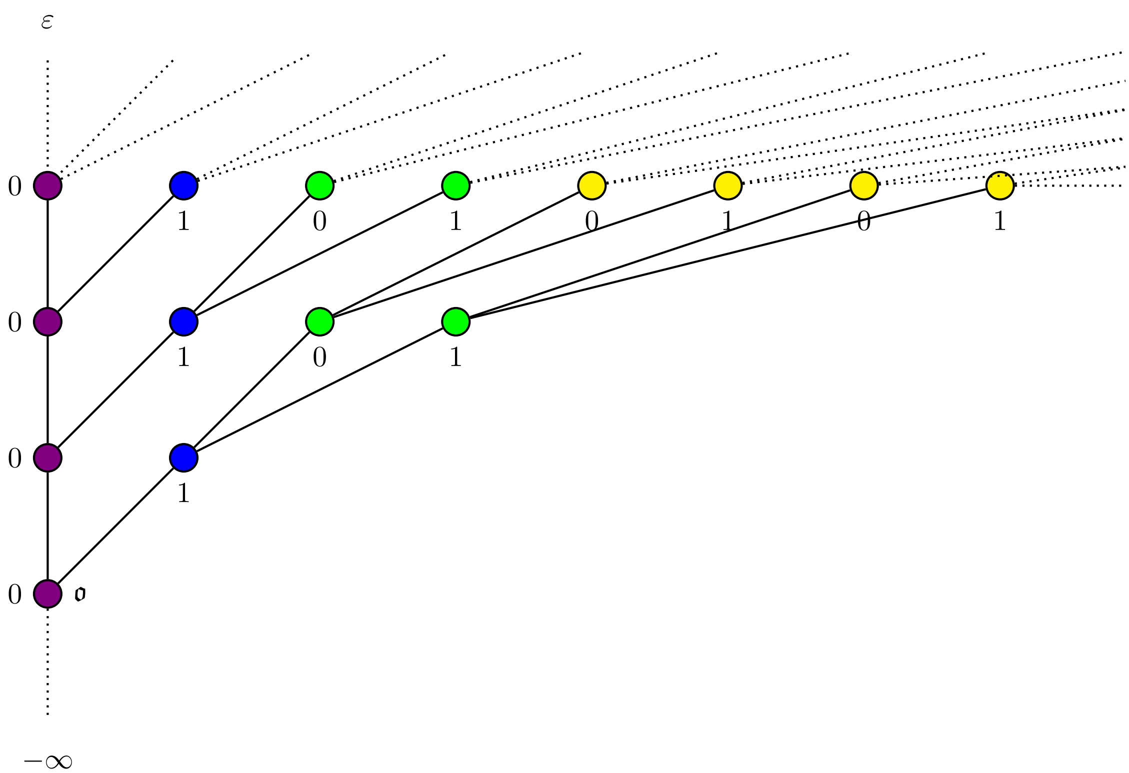

Figure 3.5: The tree of Figure 3.4, organized so each row is a horocycle

of x such that $\mathfrak{h}(x) < \mathfrak{h}(y)$ as *successors* of x. As the remaining neighbor z is such that $\mathfrak{h}(x) > \mathfrak{h}(y)$, we refer to z as the *predecessor* of x.

Definition 25. To each node x in $\mathbb{T}_q$, and so to each ray $\overline{x(-\infty)}$, associate the leftward sequence $(x_{-\infty}(k - \mathfrak{h}(x)))_{k \le \mathfrak{h}(x)}$. This places each the label of each node $x_{-\infty}(i)$ along $\overline{x(-\infty)}$ at index $\mathfrak{h}(x_{-\infty}(i)) = \mathfrak{h}(x) - i$.

Lemma 5. The association of Definition 25 is a bijection between $\left\{\overline{x(-\infty)} : x \in \mathbb{T}_q\right\}$ and $\bigcup\{\oplus_{k \le n}(\mathbb{Z}/q\mathbb{Z}) : n \in \mathbb{Z}\}$.

Proof. Let $x, y \in \mathbb{T}_q$ be distinct. The starting index of x's sequence is $\mathfrak{h}(x)$ and the starting index of y's is $\mathfrak{h}(y)$. If x and y do not belong to the same horocycle, then $\mathfrak{h}(x) \neq \mathfrak{h}(y)$ and thus their sequences are distinct.

Suppose henceforth that x and y do belong to the same horocycle, so $\mathfrak{h}(x) = \mathfrak{h}(y)$. By

Lemma 4, there is at most one node representing a given horocycle along a path to $-\infty$. It follows that at most one of x or y is along $\overline{\mathfrak{o}(-\infty)}$. Let $c_x = x \curlywedge \mathfrak{o}$, let $c_y = y \curlywedge \mathfrak{o}$, and consider $\overline{xc_x}$ and $\overline{yc_y}$, indexed as sub-sequences of their respective leftward sequences.

If $\overline{xc_x}$ and $\overline{xc_y}$ differ in length, then without loss of generality assume

$$n = \text{length}(\overline{xc_x}) > \text{length}(\overline{yc_y}) \geq 0.$$

Then $x \neq c_x$, so as c_x is along $\overline{\mathfrak{o}(-\infty)}$, it follows that x cannot be. In particular, $x_{-\infty}(n-1)$ is not labeled with 0 by Definition 22. Every node along $\overline{c_x(-\infty)}$ and $\overline{c_y(-\infty}$ is labeled with 0, and $y_{-\infty}(n-1)$ is along $\overline{c_y(-\infty)}$, so it follows that the label of $y_{-\infty}(n-1)$ is 0. Thus the associated infinite leftward sequences differ.

If the sequences $\overline{xc_x}$ and $\overline{xc_y}$ agree in length, then $d(c_x, x) = d(c_y, y)$. As $\mathfrak{h}(x) = \mathfrak{h}(y)$, it follows from Lemma 4 that $\mathfrak{h}(c_x) = \mathfrak{h}(c_y)$. Both c_x and c_y are along $\overline{\mathfrak{o}(-\infty)}$, so by Lemma 4 again we have $c_x = c_y$. Suppose $\text{length}(\overline{xc_x}) = n = \text{length}(\overline{yc_y})$. If the leftward sequences associated with x and y are to agree, then the label of $x_{-\infty}(n-1)$ agrees with that of $y_{-\infty}(n-1)$. This implies $x_{-\infty}(n-1) = y_{-\infty}(n-1)$, as there is a single neighbor of $c_x = c_y$ with this label. By finite induction, we may reach the same conclusion about $x_{-\infty}(n-i)$ and $y_{-\infty}(n-i)$ for each $0 \leq i \leq n$, giving $x = y$. This concludes the injective portion of the argument.

To see that the association is onto, we induct on the length of the shortest leftward prefix of $(v_k)_{k \leq n} \in \bigcup\{\oplus_{k \leq n}(\mathbb{Z}/q\mathbb{Z}) : n \in \mathbb{Z}\}$ which contains all nonzero elements in the sequence. If this length is 0, then the leftward sequence of $\mathfrak{o}_{-\infty}(-n)$ along $\overline{\varepsilon(-\infty)}$ is the leftward sequence starting at index n which is constantly 0.

Suppose that the mapping is onto for the subset of elements of $\{\oplus_{k \leq n}(\mathbb{Z}/q\mathbb{Z}) : n \in \mathbb{Z}\}$ with a length n shortest leftward prefix containing all nonzero elements. Then if $(v_k)_{k \leq n} \in \{\oplus_{k \leq n}(\mathbb{Z}/q\mathbb{Z}) : n \in \mathbb{Z}\}$ is such that this shortest prefix has length $n+1$, let $x \in \mathbb{T}_q$ be associated with $(v_k)_{k \leq n-1}$. Then as x has a successor with label v_n by Definition 22, we are done. ■

Example 12. For each node in Figure 3.5, its associated leftward sequence is given by tracing the path from the node to the origin $\mathfrak{o}$. The starting index for each sequence is given by the row on which the node is located, starting from index 0 at the bottom row. It should be noted that for all of these nodes, the portion of the sequence that is not shown is constantly 0. This is because the remainder of the associated path is along $\overline{\mathfrak{o}(-\infty)}$, and we know from Definition 22 that every node along this path is labeled with a 0.

Note that the Busemann function $\mathfrak{h}$ thaw we have been utilizing is technically the $-\infty$-Busemann function. That is, $\mathfrak{h}$ depends on the end $\curlywedge$ refers to. In a moment, we will want to use two Busemann functions that refer to distinct ends simultaneously. To avoid confusion, we will subscript $\mathfrak{h}$ by the end it considers.

Definition 26. If $\mathfrak{h}$ is defined in terms of some end ω, we, for each vertex x in $\mathbb{T}_q$, identify the path $\overline{x\omega}$ with the leftward sequence associated to x by $\mathfrak{h}$.

Definition 27. Let $\mathbb{T}_q$ and $\mathbb{T}_r$ be trees with respective roots $\mathfrak{o}_q$, $\mathfrak{o}_r$ and respective ends $-\infty$, ∞. The *Diestel-Leader graph* $\mathrm{DL}(q,r)$ is given by the vertex set

$$\{(x,y) \in \mathbb{T}_q \times \mathbb{T}_r : \mathfrak{h}_{-\infty}(x) = -\mathfrak{h}_{\infty}(y)\} = \{(x,y) \in \mathbb{T}_q \times \mathbb{T}_r : \mathfrak{h}_{-\infty}(x) = \mathfrak{m}_{\infty}(y) - 1\}$$

where $\mathfrak{m}_\infty$ is the *modified Busemann function* $\mathfrak{m}_\infty(x) = 1 - \mathfrak{h}_\infty(x)$. The graph has the edge $\{(u,v),(x,y)\}$ if and only if $\{u,x\}$ and $\{v,y\}$ are edges in $\mathbb{T}_q$ and $\mathbb{T}_r$, respectively.

Example 13. Figure 3.6 displays a picture of this graph made in Geogebra [11]. Here, segments of two 4-regular trees, embedded in $\mathbb{R}^3$ along perpendicular planes, are colored red and blue. Denote them $\mathbb{T}_r$ and $\mathbb{T}_b$. The tree $\mathbb{T}_b$ is organized into horocycles with respect the $\mathfrak{h}_{-\infty}$ Busemann function, while $\mathbb{T}_r$ is organized into horocycles with respect to the $\mathfrak{h}_\infty$ Busemann function. Moreover $\mathfrak{h}_{-\infty}(X) = -\mathfrak{h}_\infty(C) = \mathfrak{m}_\infty(C) - 1$. The segment of the Diestel-Leader graph $\mathrm{DL}(3,3)$ corresponding to the respective segments of the two trees is shown in purple. The node corresponding to a particular pair $(x,y) \in \mathbb{T}_r \times \mathbb{T}_b$ that is present in $\mathrm{DL}(3,3)$ is denoted through concatenation—for example, the node $(X,C) \in \mathrm{DL}(3,3)$ is labeled XC in the graph. It should be noted that while

DL(3, 3) appears to be acyclic from the segment shown, this is not in fact the case. For example, the node Y has two successors not shown as $\mathbb{T}_b$ is 4-regular. Let us refer to them as $X2$ and $X3$. As $\mathfrak{h}_{-\infty}(X3) = \mathfrak{h}_{-\infty}(X2) = \mathfrak{h}_{-\infty}(X)$ and $\mathfrak{h}_{-\infty}(X) = -\mathfrak{h}_{\infty}(C)$, it follows that the pairs $(X3, C)$ and $(X2, C)$ reside in DL(3, 3). As $\{X1, Y\}$, $\{X2, Y\}$ are edges in $\mathbb{T}_b$ and $\{C, D1\}$, $\{C, D2\}$ are edges $\mathbb{T}_r$ in $\mathbb{T}_b$, $\mathbb{T}_b$, and $\mathbb{T}_r$, respectively, it follows that $\{(X, C), (Y, D1)\}, \{(Y, D1), (X2, C)\}, \{(X2, C), (Y, D2)\}, \{(Y, D2), (X, C)\}$ are edges in DL(3, 3). This cycle is shown and bolded in Figure 3.7 and in this graph. This graph shows a highly cyclic segment of DL(2, 2). For additional, more comprehensive depictions of Diestel-Leader graphs, see [6].

Definition 28. In the tree $\mathbb{T}_r$ of Definition 27, re-index each $\overline{y\infty}$ so that the entry $y_\infty(i)$ is located at index $\mathfrak{m}(y_\infty(i))$, its modified Busemann function value. By Lemma 4 and the definition of the Diestel-Leader graph, it follows that the sequences of x and y can be concatenated in place (without re-indexing) to make a bi-infinite sequence. Once concatenated, however, we lose the ability to retrieve the values $\mathfrak{h}_{-\infty}(x)$ and $\mathfrak{m}_\infty(y) = 1 - \mathfrak{h}_\infty(y) = \mathfrak{h}_{-\infty}(x) + 1$, as these were stored as the starting indices of the two sequences. To remedy this, we tag the sequence by pairing it with $\mathfrak{h}(x)$. By Lemma 5, there is a bijective correspondence between $(\oplus_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z})) \times \mathbb{Z}$ and DL(q, q).

Let G be a group with $|G| = q$. By Definition 28, indexing G by $\{0, \ldots, q-1\}$ allows a bijective correspondence between DL(q, q) and L_G. If we view each vertex of DL(q, q) as an element of L_G, then the resulting action of L_G on DL(q, q) is transitive and has no fixed points. If we can find a collection $\ell_1, \ldots, \ell_n \in L_G$ for which the neighbors $\{v_k\}_k$ of some node $v \in$ DL(q, q) are all given by $\ell_i \cdot v$ for appropriate i, then it follows that DL(q, q) is a Cayley graph of L_G.

Theorem 4. Fix a positive integer q and let G be an order q group. Then the collection $\{(g^{\delta_{1j}})_j \times 1 \in L_G : g \in G\}$ generates the lamplighter group L_G, and there is a labeling of the nodes and edges of the Diestel-Leader graph $DL(q, q)$ which makes it the Cayley graph of L_G under the generating set .

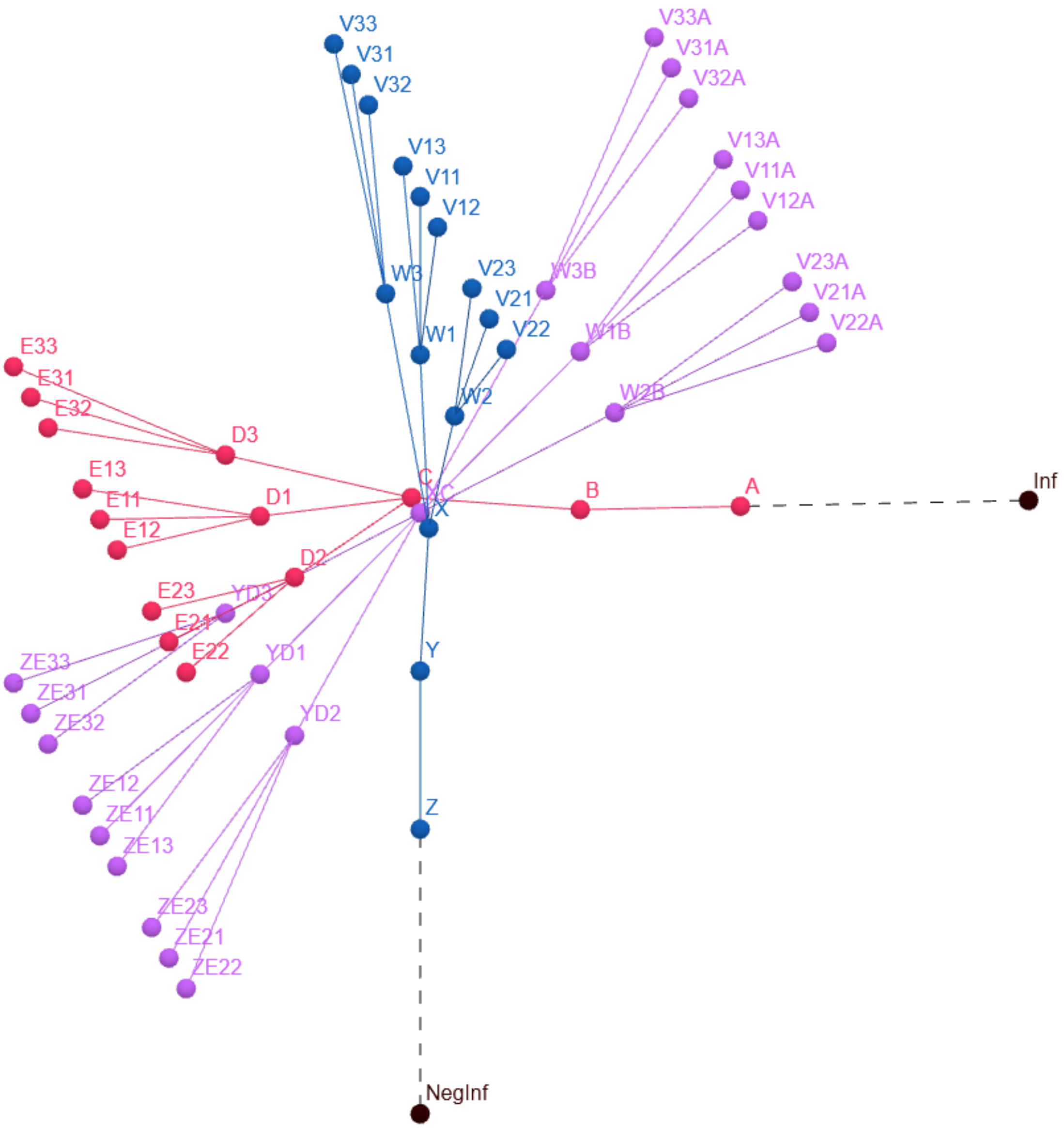

Figure 3.6: Segments of two 4-regular trees in red and blue and the corresponding segment of DL(3, 3) in purple—see Example 13

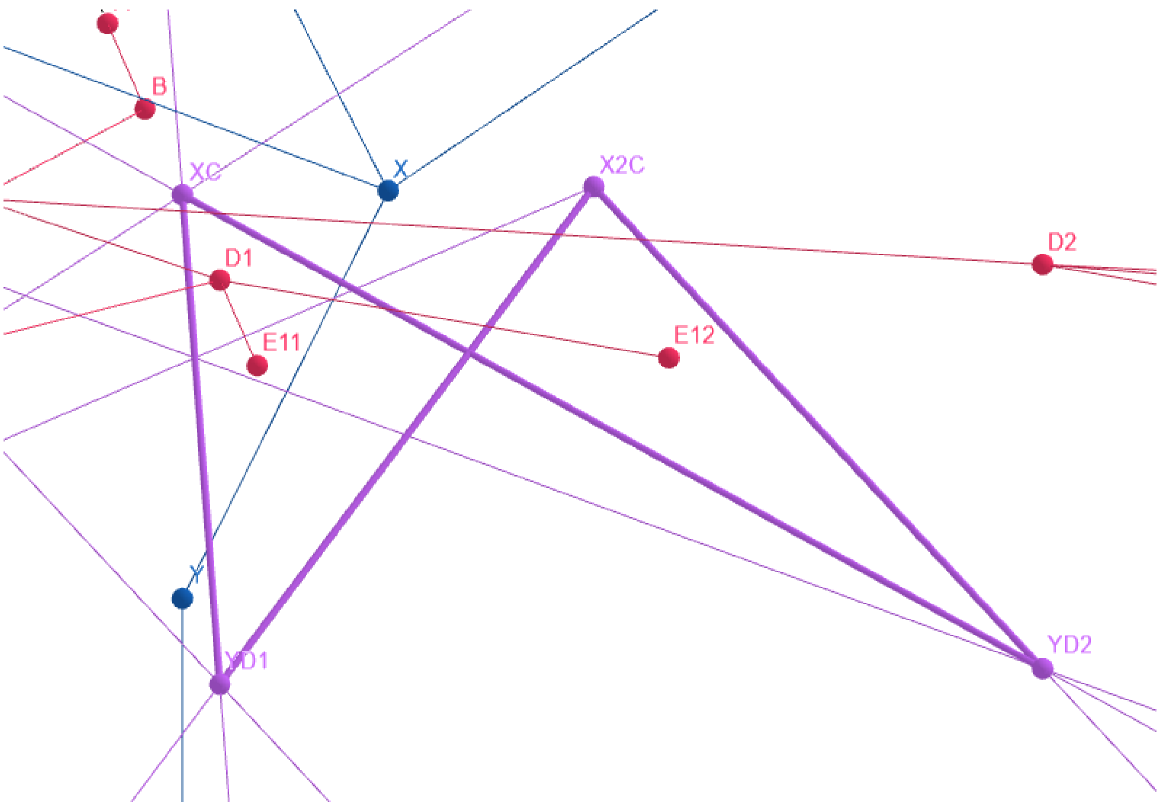

Figure 3.7: The DL(3, 3) graph as presented in Figure 3.6, with additional nodes and a bolded cycle as described in Example 13

Proof. We first show that the collection $\{(g^{\delta_{ij}})_j \times 1 \in L_G : g \in G\}$ generates L_G. Using that $(\vec{1},1)^{-1} = (\vec{1},-1)$, compute

$$(\vec{1},1)^{-1}((g^{\delta_{1j}})_j,1) = (\vec{1},-1)((g^{\delta_{1j}})_j,1) = ((g^{\delta_{0j}})_j,0).$$

As the collection generates the generators used in Theorem 3, it in turn generates L_G.

Identify G with $\{0,\ldots,q-1\}$ under any bijection, and let $\dot{a}$ denote the label of the vertex a in $\mathbb{T}_q$. Let $b \times \beta \in \mathrm{DL}(q,q)$ with $\mathfrak{h}(b) = m$. Necessarily, $\mathfrak{m}(\beta) = m+1$ and $b \times \beta$ corresponds to some tagged sequence

$$\left(\begin{pmatrix} \ldots, & m-1, & m, & m+1, & m+2, & \ldots \\ \ldots, & \dot{a}, & \dot{b}, & \dot{\beta}, & \dot{\alpha}, & \ldots \end{pmatrix}, m\right),$$

where the top row denotes the indices and the bottom row denotes the entries. If the successors of b are $c_0,\ldots,c_{q-1}$, then the $\mathrm{DL}(q,q)$-neighbors $c_k \times \alpha$ of $b \times \beta$ correspond to

$$\left(\begin{pmatrix} \ldots, & m-1, & m, & m+1, & m+2, & \ldots \\ \ldots, & \dot{a}, & \dot{b}, & \dot{c_k}, & \dot{\alpha}, & \ldots \end{pmatrix}, m+1\right)$$

which in L_G is equal to the product

$$\left(\begin{pmatrix} \ldots, & m-1, & m, & m+1, & m+2, & \ldots \\ \ldots, & \dot{a}, & \dot{b}, & \dot{\beta}, & \dot{\alpha}, & \ldots \end{pmatrix}, m\right) \cdot \left(((\dot{\beta}^{-1}\dot{c_k})^{\delta_{1j}})_j, 1\right).$$

If the successors of β are $\gamma_0,\ldots,\gamma_{q-1}$, the $\mathrm{DL}(q,q)$-neighbors $a \times \gamma_k$ of $b \times \beta$ correspond to

$$\left(\begin{pmatrix} \ldots, & m-1, & m, & m+1, & m+2, & \ldots \\ \ldots, & \dot{a}, & \dot{\gamma_k}, & \dot{\beta}, & \dot{\alpha}, & \ldots \end{pmatrix}, m-1\right)$$

which in L_G is equal to the product

$$\left(\begin{pmatrix} \dots, & m-1, & m, & m+1, & m+2, & \dots \\ \dots, & \dot{a}, & \dot{b}, & \dot{\beta}, & \dot{\alpha}, & \dots \end{pmatrix}, m\right) \cdot \left(((\dot{b}^{-1}\dot{\gamma_k})^{\delta_{0j}})_j, -1\right)$$

as $\left((\dot{b}^{-1}\dot{\gamma_k})_{0j}, -1\right) = \left((\dot{\gamma_k}^{-1}\dot{b})_{1j}, 1\right)^{-1}$. Thus there is a 1-1 correspondence between the neighbors of a node in $\mathrm{DL}(q, q)$ and the generators (and their inverses) chosen for L_G. ∎

3.3 The Traditional Lamplighter is an Automaton Group

This proof is an elaboration on the one presented in [13]. The aim is to relate the group generated by Figure 1.2 to a group of functions that act on power series in $(\mathbb{Z}/2\mathbb{Z})[\![t]\!]$, and then to show that this second group is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \,\mathrm{wr}\, \mathbb{Z}$. We begin by recalling the following lemma:

> **Lemma** 3. Let $G = \langle u, v\rangle$, where $v^2 = 1$. Then every element $g \in G$ is such that $g = ({}^{u^{\sigma_1}}v) \cdots ({}^{u^{\sigma_n}}v)u^{-\sigma_n}$ for appropriate $\sigma_1, \dots, \sigma_n \in \mathbb{Z}$.

The group of transforms we aim to use is generated by two functions, namely

$$(\mathbb{Z}/2\mathbb{Z})[\![t]\!] \to (\mathbb{Z}/2\mathbb{Z})[\![t]\!] : f(t) \mapsto f(t) + 1, \qquad (\mathbb{Z}/2\mathbb{Z})[\![t]\!] \to (\mathbb{Z}/2\mathbb{Z})[\![t]\!] : f(t) \mapsto (1+t)f(t).$$

Note that the first function is its own inverse.

Lemma 6. Let $\phi_\tau, \phi_b : (\mathbb{Z}/2\mathbb{Z})[\![t]\!] \to (\mathbb{Z}/2\mathbb{Z})[\![t]\!]$ be given by $\phi_\tau(f(t)) = f(t) + 1$ and $\phi_b(f(t)) = (1+t)f(t)$. Then $\langle \phi_\tau, \phi_b\rangle$ contains the collection

$$\mathscr{C} = \left\{ f(t) \mapsto (1+t)^n f(t) + \sum_{s\in\mathbb{Z}} B_s(1+t)^s : n \in \mathbb{Z},\, (B_s)_s \in \oplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) \right\}.$$

Proof. Let $\phi \in \mathscr{C}$ be given by

$$\phi(f(t)) = (1+t)^{\ell_0} f(t) + \sum_{s=1}^{\tilde{s}} (1+t)^{\ell_s}.$$

Then we can compute

$$\begin{aligned}
\phi(f(t)) &= (1+t)^{\ell_0} f(t) + \sum_{s=1}^{\tilde{s}} (1+t)^{\ell_s} \\
&= (1+t)^{\ell_{\tilde{s}}} \left((1+t)^{-\ell_{\tilde{s}}} \left((1+t)^{\ell_0} f(t) + \sum_{s=1}^{\tilde{s}-1} (1+t)^{\ell_s} \right) + 1 \right) \\
&= \left({}^{\phi_b^{\ell_{\tilde{s}}}} \phi_\tau \right) \left((1+t)^{\ell_0} f(t) + \sum_{s=1}^{\tilde{s}-1} (1+t)^{\ell_s} \right).
\end{aligned}$$

By induction

$$\begin{aligned}
\phi(f(t)) &= (1+t)^{\ell_0} f(t) + \sum_{s=1}^{\tilde{s}} (1+t)^{\ell_s} \\
&= \left({}^{\phi_b^{\ell_{\tilde{s}}}} \phi_\tau \circ \cdots \circ {}^{\phi_b^{\ell_1}} \phi_\tau \right) (1+t)^{\ell_0} f(t) \\
&= \left({}^{\phi_b^{\ell_{\tilde{s}}}} \phi_\tau \circ \cdots \circ {}^{\phi_b^{\ell_1}} \phi_\tau \circ \phi_b^{\ell_0} \right) f(t).
\end{aligned}$$

Thus ϕ resides in $\langle \phi_\tau, \phi_b \rangle$, so $\langle \phi_\tau, \phi_b \rangle$ contains $\mathscr{C}$. ■

To show that the reverse containment holds, As $\phi_\tau, \phi_b \in \mathscr{C}$, it suffices to show that $\mathscr{C}$ is a group under composition. In fact:

Lemma 7. The collection $\mathscr{C}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \operatorname{wr} \mathbb{Z}$.

Proof. There is a 1-1 correspondence between $\mathscr{C}$ and $\oplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$ by

$$\left(f(t) \mapsto (1+t)^n f(t) + \sum_{s \in \mathbb{Z}} B_s (1+t)^s \right) \leftrightarrow ((B_s)_s, n).$$

By Lemma 6, $\mathscr{C} \subseteq \langle \phi_\tau, \phi_b \rangle$, so by Lemma 3 we can write every element of $\mathscr{C}$ as a product of form ${}^{\phi_b^{\ell_{\tilde{s}}}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{\ell_1}}\phi_\tau \circ \phi_b^{\ell_0}$. Critically,

$$ {}^{\phi_b^{\ell_{\tilde{s}}}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{\ell_1}}\phi_\tau \circ \phi_b^{\ell_0} = \phi_b^{\ell_0} \circ {}^{\phi_b^{\ell_{\tilde{s}}-\ell_0}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{\ell_1-\ell_0}}\phi_\tau, $$

so if ϕ and ψ are in $\mathscr{C}$ by $\phi = {}^{\phi_b^{\ell_{\tilde{s}}}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{\ell_1}}\phi_\tau \circ \phi_b^{\ell_0}$ and $\psi = {}^{\phi_b^{k_{\tilde{t}}}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{k_1}}\phi_\tau \circ \phi_b^{k_0}$, then

$$ \begin{aligned} \phi \circ \psi &= \left({}^{\phi_b^{\ell_{\tilde{s}}}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{\ell_1}}\phi_\tau \circ \phi_b^{\ell_0}\right) \circ \left({}^{\phi_b^{k_{\tilde{t}}}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{k_1}}\phi_\tau \circ \phi_b^{k_0}\right) \\ &= \left({}^{\phi_b^{\ell_{\tilde{s}}}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{\ell_1}}\phi_\tau\right) \circ \left(\phi_b^{\ell_0} \circ {}^{\phi_b^{k_{\tilde{t}}}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{k_1}}\phi_\tau \circ \phi_b^{k_0}\right) \\ &= \left({}^{\phi_b^{\ell_{\tilde{s}}}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{\ell_1}}\phi_\tau\right) \circ \left({}^{\phi_b^{k_{\tilde{t}}+\ell_0}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{k_1+\ell_0}}\phi_\tau \circ \phi_b^{\ell_0+k_0}\right) \\ &= {}^{\phi_b^{\ell_{\tilde{s}}}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{\ell_1}}\phi_\tau \circ {}^{\phi_b^{k_{\tilde{t}}+\ell_0}}\phi_\tau \circ \cdots \circ {}^{\phi_b^{k_1+\ell_0}}\phi_\tau \circ \phi_b^{\ell_0+k_0}. \end{aligned} $$

Let $(B_s)_s = \sum_{i=1}^{\tilde{s}} (\delta_{\ell_i,j})_j$ and let $(A_s)_s = \sum_{i=1}^{\tilde{t}} (\delta_{k_i,j})_j$. Then ϕ is identified with $((B_s)_s, \ell_0)$ and ψ is identified with $((A_s)_s, k_0)$. The previous computation in tandem with the work of Lemma 6 shows that

$$ ((B_s)_s, \ell_0)((A_s)_s, k_0) = ((B_s + A_{s-\ell_0})_s, \ell_0 + k_0), $$

so $\langle \phi_\tau, \phi_b \rangle$ is isomorphic to $\oplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z} = (\mathbb{Z}/2\mathbb{Z}) \operatorname{wr} \mathbb{Z}$. ■

Thus we obtain

Corollary 2. Let $\phi_\tau, \phi_b : (\mathbb{Z}/2\mathbb{Z})[\![t]\!] \to (\mathbb{Z}/2\mathbb{Z})[\![t]\!]$ be given by $\phi_\tau(f(t)) = f(t) + 1$ and $\phi_b(f(t)) = (1+t)f(t)$. Then $\langle \phi_\tau, \phi_b \rangle \cong (\mathbb{Z}/2\mathbb{Z}) \operatorname{wr} \mathbb{Z}$.

We use this to derive the desired theorem:

Theorem 5. The traditional lamplighter group L_2 is the group generated by the automaton in Figure 1.2, formally defined to be

$$\mathcal{L}_2 = \left(A = \{0,1\}, Q = \{a,b\}, \left\{ \begin{array}{ccc} (q,0) & \mapsto & b \\ (q,1) & \mapsto & a \end{array} \right\}, \left\{ \begin{array}{ccc} (a,x) & \mapsto & (0\,1)x \\ (b,x) & \mapsto & x \end{array} \right\} \right).$$

Proof. Let $\tau = (0\,1)$. One should immediately note that b^{-1}, given by some $(\sigma, (c,d)) \in S_A \wr \operatorname{Aut} A^{\leq \omega}$ due to Theorem 1, satisfies

$$bb^{-1} = (1\sigma, (b|_{\sigma 0} c, a|_{\sigma 1} d)) = (1, (\mathrm{Id}_A, \mathrm{Id}_A)).$$

Then $1 = 1\sigma = \sigma$, so $b|_{\sigma 0} = b|_0 = b$ and $a|_{\sigma 1} = a|_1 = a$. Thus $c = b^{-1}$ and $d = a^{-1}$, so $\tau = ab^{-1}$ and $\langle a, b\rangle = \langle \tau, b\rangle$.

Identify A with $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Automata are closed under their transition functions, so if $\{\xi_i\}_{i\in\mathbb{N}_+}$ is a subset of A and n is a positive natural number, then $b|_{\xi_1\cdots\xi_{n-1}}$ is either a or b. If

$$b(\xi_1\xi_2\cdots\xi_n) = \xi_1^\frown(\xi_1+\xi_2)^\frown\cdots^\frown(\xi_{n-1}+\xi_n)$$

and $\xi_n = 0$, then $b|_{\xi_1\cdots\xi_n} = b$, so

$$b|_{\xi_1\cdots\xi_n}(\xi_{n+1}) = b(\xi_{n+1}) = \xi_{n+1} = 0 + \xi_{n+1} = \xi_n + \xi.$$

If instead $\xi_n = 1$, then $b|_{\xi_1\cdots\xi_n} = a$ and

$$b|_{\xi_1\cdots\xi_n}(\xi_{n+1}) = a(\xi_{n+1}) = \tau(\xi_{n+1}) = 1 + \xi_{n+1} = \xi_n + \xi_{n+1}.$$

Thus the action of b can be written as $b(\xi_1\xi_2\xi_3\cdots) = \xi_1^\frown(\xi_1+\xi_2)^\frown(\xi_2+\xi_3)^\frown\cdots$. The action of τ can be written as $\tau(\xi^\frown x) = (\xi+1)^\frown x$.

Identify $(\xi_n)_{n\in\mathbb{N}} \in X^\omega$ with the formal power series $f(t) = \sum_{i=0}^{\infty} \xi_i t^i$ in $\mathbb{F}_2[\![t]\!]$. Under this identification, τ maps to $\phi_\tau : f(t) \mapsto f(t) + 1$ and b maps to $\phi_b : f(t) \mapsto (1+t)f(t)$. As $\langle \phi_\tau, \phi_b\rangle \cong \langle \tau, b\rangle \cong \langle a, b\rangle$, to show that $(\mathbb{Z}/2\mathbb{Z}) \operatorname{wr} \mathbb{Z}$ is generated by $\mathcal{A}$, it suffices to show that $\langle \phi_\tau, \phi_b\rangle \cong (\mathbb{Z}/2\mathbb{Z}) \operatorname{wr} \mathbb{Z}$. But this the result of Corollary 2. ■

3.4 When is a Lamplighter Group an Automaton Group?

We have just seen that the traditional lamplighter group L_2 is an automaton group. It is natural to ask if the generalizations can also be generated by automata.

3.4.1 Case 1: Finite Abelian Underlying Group

One might be unsurprised to learn that for any finite cyclic group $\mathbb{Z}/n\mathbb{Z}$, the corresponding lamplighter group $L_n = (\mathbb{Z}/n\mathbb{Z}) \operatorname{wr} \mathbb{Z}$ is also an automaton group. In fact, given any finite Abelian group $G = (\mathbb{Z}/n_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/n_m\mathbb{Z})$, we can find a finite state automaton that generates it.

We start with a generalization of Lemma 3:

Lemma 8. Let G be a group, let $H \leq G$, and let $a \in G$ be such that $G = \langle H, a\rangle$. Then the normal closure $\operatorname{ncl}(H)$ equals $\langle {}^{a^m}h : h \in H, m \in \mathbb{Z}\rangle$ (where ${}^{a^m}h = a^m h a^{-m}$), and $G = (\operatorname{ncl}(H))\langle a\rangle$.

Proof. First, $\langle {}^{a^m}h : h \in H, m \in \mathbb{Z}\rangle$ is a subgroup of $\operatorname{ncl}(H)$ as $\langle a\rangle \leq G$ and $\operatorname{ncl}(H) = \langle {}^{g}h : h \in H, g \in G\rangle$. If $h_1, \ldots, h_n \in H$ and $\ell_1, \ldots, \ell_n \in \mathbb{Z}$, then

$$a^{\ell_1}h_1 \cdots a^{\ell_n}h_n = {}^{a^{\sigma_1}}h_1 \cdots {}^{a^{\sigma_n}}h_n a^{\sigma_n},$$

where $\sigma_i = \sum_{j=1}^{i} \ell_i$ for each i. Similarly,

$$h_n^{-1}a^{-\ell_n} \cdots h_1^{-1}a^{-\ell_1} = (a^{-\sigma_n})({}^{a^{\sigma_n}}h_n^{-1}) \cdots ({}^{a^{\sigma_1}}h_1^{-1})$$

(note $({}^{a^{\sigma_i}}h_1)^{-1}$ and ${}^{a^{\sigma_i}}(h_1^{-1})$ are equal). As every $g \in G = \langle H, a\rangle$ has form $h_1a^{\ell_1} \cdots h_na^{\ell_n}$ for the appropriate $h_1, \ldots, h_n \in H$ and $\ell_1, \cdots \ell_n \in \mathbb{Z}$,

$$(a^{\ell_1}h_1 \cdots a^{\ell_n}h_n)h(h_n^{-1}a^{-\ell_n} \cdots h_1^{-1}a^{-\ell_1}) = ({}^{\sigma_1}h_1 \cdots {}^{\sigma_n}h_n)^{\sigma_n}h\big({}^{\sigma_n}h_n^{-1} \cdots {}^{\sigma_1}h_1^{-1}\big).$$

Thus $\operatorname{ncl}(H) \leq \langle {}^{a^m}h : h \in H, m \in \mathbb{Z}\rangle$, so equality holds.

As already noted, every $g \in G$ has form $g = h_1 a^{\ell_1} \cdots h_n a^{\ell_n} = {}^{a^{\sigma_1}}h_1 \cdots {}^{a^{\sigma_n}}h_n a^{\sigma_n}$ and so is in $(\mathrm{ncl}(H))\langle a \rangle$. As $\mathrm{ncl}(H)$ and $\langle a \rangle$ are subgroups of G it follows that $(\mathrm{ncl}(H))\langle a \rangle = G$. ■

To prove that L_G is generated by an automaton when G is Abelian, we use a collection of automata referred to as the Cayley machines. The definition for this collection was given in Example 7 and is restated here:

Definition 29. Let G be a finite group. Its *Cayley Machine*, denoted $\mathcal{C}(G)$, is the automaton with alphabet G, collection of states G, output function $\lambda(g_1, g_2) = g_1 g_2$, and transition function $\delta(g_1, g_2) = g_1 g_2$. In the Moore diagram of $\mathcal{C}(G)$, there is an edge between every pair of nodes, and the edge from the state g_1 to the state g_2 has label $g_1^{-1} g_2 \mid g_2$. See Figure 2.5 for the Cayley machine of the dihedral group $D_{2 \cdot 3}$.

Remark 4. Because the elements of G are already members of a group, $\hat{g}$ is used refer to g as an element of $\mathrm{Aut}\, G^{\leq \omega}$. Using the result of Theorem 1, if $G = \{g_1 = 1, \ldots, g_n\}$ and $g \in G$, then $\hat{g} = (g(\,\cdot\,), (\widehat{g_1 g}, \ldots, \widehat{g_n g}))$ and $\hat{g}^{-1} = \big(g^{-1}(\,\cdot\,), \big(\hat{g_1}^{-1}, \ldots, \hat{g_n}^{-1}\big)\big)$. That is, on input $h \in G$, the state $\hat{g} \in \mathcal{C}(G)$ outputs gh and goes to state $\widehat{gh}$, while the state $\hat{g}^{-1} \in \mathcal{C}(G)^{-1}$ outputs $g^{-1}h$ and goes to state $\hat{h}^{-1}$. From this, we can derive the product that results from left-multiplying an element $\hat{g}$ from $\mathcal{C}(G)$ by an element $\hat{h}^{-1}$ from $\mathcal{C}(G)^{-1}$:

$$\hat{h}^{-1}\hat{g} = (h^{-1}, (\hat{g_1}^{-1}, \ldots, \hat{g_n}^{-1}))(g, (\widehat{g_1 g}, \ldots, \widehat{g_n g}))$$

$$= (h^{-1}g, (\widehat{g_1 g}^{-1}\widehat{g_1 g}, \ldots, \widehat{g_n g}^{-1}\widehat{g_1 g})$$

$$= (h^{-1}g, (\mathrm{Id}_{G^{\leq \omega}}, \ldots, \mathrm{Id}_{G^{\leq \omega}}))$$

When G is a finite Abelian group, its Cayley machine $\mathcal{C}(G)$ generates a group isomorphic to the lamplighter group $L_G = G \,\mathrm{wr}\, \mathbb{Z}$. This proof, from [17], relies on a correspondence between elements of $\mathsf{G}(\mathcal{C}(G))$ and transformations of $G[\![t]\!]$, the collection of power series with coefficients in G. In this way, it is reminiscent of Theorem 5.

Lemma 9. Let G be a non-trivial finite Abelian group, and let $f(t) \in G[\![t]\!]$. Then as functions on $G[\![t]\!] \cong G^\omega$:

1. $\hat{g} : f(t) \mapsto (g + f(t))/(1-t)$.

2. $\hat{g}^{-1} : f(t) \mapsto (1-t)f(t) - g$.

Proof. Note that Item 2 is a consequence of Item 1 since $(\hat{g}^{-1}\hat{g})(f(t)) = f(t)$. That Item 1 holds is because $\hat{g} : (g_i)_{i\in\mathbb{N}} \mapsto (g + g_0 + \cdots + g_i)_{i\in\mathbb{N}}$ and $(g + g_0 + \cdots + g_i)_{i\in\mathbb{N}}$ corresponds to $\sum_{i=0}^{\infty}(g + g_0 + \cdots + g_i)t^i$ in $G[\![t]\!]$. Index g as $g = g_{-1}$, and compute

$$\begin{aligned}
(1-t)\hat{g}(f(t)) &= (1-t)\sum_{i=0}^{\infty}(g + g_0 + \cdots + g_i)t^i \\
&= (1-t)\sum_{j=-1}^{\infty}\sum_{i=j\vee 0}^{\infty} g_j t^i \\
&= \sum_{j=-1}^{\infty}\sum_{i=j\vee 0}^{\infty} g_j t^i - \sum_{j=-1}^{\infty}\sum_{i=j\vee 0}^{\infty} g_j t^{i+1} \\
&= g + \sum_{j=0}^{\infty} g_j t^j \\
&= g + f(t),
\end{aligned}$$

where $j \vee 0 = \max\{j, 0\}$. Dividing by $(1-t)$ gives that $\hat{g}(f(t)) = (g + f(t))/(1-t)$. ■

Lemma 10. Let G be a non-trivial finite Abelian group, let $a = (\hat{0})^{-1}$, and let $f(t) \in G[\![t]\!]$. Then

1. $a^n(f(t)) = (1-t)^n f(t)$ for any $n \in \mathbb{Z}$.

2. $(a\hat{g})(f(t)) = g + f(t)$ for any $g \in G$.

Proof. Item 1 is by repeated applications of Item 1 and Item 2 of Lemma 9, and Item 2 is because $a\hat{g} = g(\mathrm{Id}_{G^\omega}, \ldots, \mathrm{Id}_{G^\omega})$ by Remark 4. ■

Corollary 3. A finite group G imbeds in the group generated by its Caylely machine $\mathsf{G}(\mathcal{C}(G))$ by the map $g \mapsto a\hat{g}$, and $\mathsf{G}(\mathcal{C}(G)) = \langle G, a\rangle$ under the imbedding.

Proof. Let $\phi : G[\![t]\!] \to G[\![t]\!]$ be given by $\phi(f(t)) = g_1 + f(t)$, and let $\psi : G[\![t]\!] \to G[\![t]\!]$ be given by $\psi(f(t)) = g_2 + f(t)$. If $\phi = \psi$, then if $\vec{0}$ denotes the constant 0 function it follows $g_1 + \vec{0} = g_2 + \vec{0}$, so $g_1 = g_2$. Therefore $g \mapsto a\hat{g}$ is an injection by Lemma 10. This injection is a homomorphism as

$$(f(t) \mapsto g_1 + f(t)) \circ (f(t) \mapsto g_2 + f(t)) = (f(t) \mapsto (g_1 + (g_2 + f(t)))) = (f(t) \mapsto (g_1 + g_2) + f(t)).$$

Index G so that $G = \{g_1, \ldots, g_n\}$. As $a\hat{g} = (g, \mathrm{Id}_{G^{\leq\omega}}, \ldots, \mathrm{Id}_{G^{\leq\omega}})$ and $a^{-1} = \hat{0} = (0, (\widehat{g_1 g}, \ldots, \widehat{g_n g}))$, it follows $\hat{g} = a\hat{g}a^{-1}$, so every generator from $\mathcal{C}(G)$ is in $\langle G, a\rangle$. ■

Lemma 11. Let G be a non-trivial finite Abelian group, let $a = (\hat{0})^{-1}$, and let $f(t) \in G[\![t]\!]$. Then $({}^{a^n}\hat{g})(f(t)) = g(1-t)^n + f(t)$ for any $g \in G$ and $n \in \mathbb{Z}$.

Proof. Compute $({}^{a^n}\hat{g})(f(t)) = (a^n g)(f(t)/(1-t)^n) = g + f(t)/(1-t)^n = g(1-t)^n + f(t)$. ■

Corollary 4. Let G be a finite Abelian group, embedded in the group generated by its Cayley machine $\mathsf{G}(\mathcal{C}(G))$. Then $\mathrm{ncl}(G) = \left\langle {}^{a^m}\hat{g} : g \in G, m \in \mathbb{N} \right\rangle$ is isomorphic to $\oplus_{\mathbb{Z}} G$

Proof. The isomorphism is given by mapping ${}^{a^m}\hat{g} = f(t) \mapsto g(1-t)^m + f(t)$ to $(g\delta_{mj})_{j\in\mathbb{Z}}$. ■

Thus we arrive at

Theorem 6. Let G be a non-trivial finite Abelian group. Then $\mathsf{G}(\mathcal{C}(G)) \cong G \operatorname{wr} \mathbb{Z}$.

Proof. By Corollary 3 and Lemma 8, we have that $\mathsf{G}(\mathcal{C}(G)) = (\mathrm{ncl}\, G)\langle a\rangle$. By Item 1 of Lemma 10 the order of a is infinite, so by Corollary 4 we can write $\mathsf{G}(\mathcal{C}(G)) = (\oplus_{\mathbb{Z}} G)\mathbb{Z}$. As every element of $\oplus_{\mathbb{Z}} G$ has finite order, it follows $\mathsf{G}(\mathcal{C}(G)) = (\oplus_{\mathbb{Z}} G) \rtimes \mathbb{Z} = G \operatorname{wr} \mathbb{Z}$. ■

3.4.2 Case 2: Finite Non-Abelian Underlying Group

Given the result of Section 3.4.1, it is somewhat surprising that if G is non-Abelian, then $G \,\mathrm{wr}\, \mathbb{Z}$ is not even a self-similar group. The proofs given here—save Theorem 8—are extracted and adapted from [10], and the line of reasoning is suggested by [19].

The proof begins with two rather computational lemmas from [10].

Lemma 12. If G is a group, then $\oplus_{\mathbb{Z}}[G,G] = [\oplus_{\mathbb{Z}}G, \oplus_{\mathbb{Z}}G]$ and

$$([g,h]^{\delta_{ij}})_j = [(g^{\delta_{ij}})_j, (h^{\delta_{ij}})_j].$$

Proof. Let $g, h \in G$ and let $i \in \mathbb{Z}$. Then

$$\begin{aligned}
([g,h]^{\delta_{ij}})_j &= ((g^{-1}h^{-1}gh)^{\delta_{ij}})_j \\
&= (g^{-\delta_{ij}}h^{-\delta_{ij}}g^{\delta_{ij}}h^{\delta_{ij}})_j \\
&= (g^{-\delta_{ij}})_j(h^{-\delta_{ij}})_j(g^{\delta_{ij}})_j(h^{\delta_{ij}})_j \\
&= (g^{\delta_{ij}})_j^{-1}(h^{\delta_{ij}})_j^{-1}(g^{\delta_{ij}})_j(h^{\delta_{ij}})_j \\
&= [(g^{\delta_{ij}})_j, (h^{\delta_{ij}})_j]
\end{aligned}$$

Thus the sets $\{([g,h]^{\delta_{ij}})_j : g,h \in G, i \in \mathbb{Z}\}$ and $\{[(g^{\delta_{ij}})_j, (h^{\delta_{ij}})_j] : g,h \in G, i \in \mathbb{Z}\}$ are equal. As these collections generate $\oplus_{\mathbb{Z}}[G,G]$ and $[\oplus_{\mathbb{Z}}G, \oplus_{\mathbb{Z}}G]$, respectively, it follows that they are equal. ■

Lemma 13. Let G be a group and let $N \trianglelefteq G \,\mathrm{wr}\, \mathbb{Z}$ intersect non-trivially with $\mathbb{Z} \leq G \,\mathrm{wr}\, \mathbb{Z}$. Then $[\oplus_{\mathbb{Z}}G, \oplus_{\mathbb{Z}}G] \leq N$.

Proof. By Lemma 12 it suffices to show that $([g,h]^{\delta_{ij}})_j \in N$ holds for all $g, h \in G$ and $i \in \mathbb{Z}$. Let $n \in (N \cap \mathbb{Z}) - \{0\}$. Then ${}^{(-n)}((g^{\delta_{ij}})_j)$ is congruent to $(g^{\delta_{ij}})_j$ modulo N as

$$((g^{\delta_{ij}})_j)^{-1} \cdot ((-n) \cdot (g^{\delta_{ij}})_j \cdot n) = (((g^{\delta_{ij}})_j)^{-1} \cdot (-n) \cdot (g^{\delta_{ij}})_j) \cdot n$$

is in N. From this it follows that $({}^{(-n)}((g^{\delta_{ij}})_j))^{-1}$ is congruent to $((g^{\delta_{ij}})_j)^{-1}$ modulo N. Lastly, as every element of $\oplus_{\mathbb{Z}}G$ is congruent to itself modulo N, we can compose the congruences to compute

$$\begin{aligned}[{}^{(-n)}(g^{\delta_{ij}})_j, (h^{\delta_{ij}})_j] &= ({}^{(-n)}(g^{\delta_{ij}})_j)^{-1} \cdot ((h^{\delta_{ij}})_j)^{-1} \cdot {}^{(-n)}(g^{\delta_{ij}})_j \cdot (h^{\delta_{ij}})_j \\ &\equiv (((g^{\delta_{ij}})_j)^{-1} \cdot ((h^{\delta_{ij}})_j)^{-1} \cdot (g^{\delta_{ij}})_j \cdot (h^{\delta_{ij}})_j) \pmod N \\ &= [(g^{\delta_{ij}})_j, (h^{\delta_{ij}})_j].\end{aligned}$$

As $([g,h]^{\delta_{ij}})_j = [(g^{\delta_{ij}})_j, (h^{\delta_{ij}})_j]$ by Lemma 12, we have

$$([g,h]^{\delta_{ij}})_j \equiv [{}^{(-n)}((g^{\delta_{ij}})_j), (g^{\delta_{ij}})_j] \pmod N.$$

As $[{}^{(-n)}(g^{\delta_{ij}})_j, (h^{\delta_{ij}})_j] = [(g^{\delta_{i+n,j}})_j, (h^{\delta_{ij}})_j] = 1$, it follows $([g,h]^{\delta_{ij}})_j$ is congruent to 1 modulo N and so is in N. ■

Theorem 7. If $G \operatorname{wr} \mathbb{Z}$ is residually finite, then G is Abelian.

Proof. Suppose G is not Abelian, so $[G,G] - \{1\}$ is non empty. If g resides in $[G,G] - \{1\}$, then $(g^{\delta_{ij}})_j$ is in $(\oplus_{\mathbb{Z}}[G,G]) - \{1\}$. Since $G \operatorname{wr} \mathbb{Z}$ is residually finite, there is a normal subgroup N of $G \operatorname{wr} \mathbb{Z}$ for which $(G \operatorname{wr} \mathbb{Z})/N$ is finite and $(g^{\delta_{ij}})_j$ is not in N. By Lemma 13 the intersection $N \cap \mathbb{Z}$ must be trivial so, $\mathbb{Z}$ is isomorphic to $N\mathbb{Z}/N$. But this is a contradiction—as $N\mathbb{Z}/N$ is a subgroup of $(G \operatorname{wr} \mathbb{Z})/N$, it would then follow that $\mathbb{Z}$ is finite. ■

It is immediate from this theorem that if G is non-Abelian, then $G \operatorname{wr} \mathbb{Z}$ is not residually finite. The next theorem gives that *all* groups generated by invertible automata are residually finite.

Theorem 8. If $\mathcal{A} = (A, Q, \lambda, \delta)$ is an invertible automaton (not necessarily finite-state), then $\mathsf{G}(\mathcal{A})$ is residually finite.

Proof. For each $n \in \mathbb{N}$, let $\operatorname{Stab}_{\mathsf{G}(\mathcal{A})}(A^n) = \{g \in \mathsf{G}(\mathcal{A}) : \forall v \in A^n(g(v) = v)\}$. Fix $n \in \mathbb{N}$, take $g \in \operatorname{Stab}_{\mathsf{G}(\mathcal{A})}(A^n)$, take $h \in \mathsf{G}(\mathcal{A})$, and take $v \in A^n$. As $|h(v)|$ is in A^n it follows that $(hgh^{-1})(v) = v$. Thus the every $\operatorname{Stab}_{\mathsf{G}(\mathcal{A})}(A^n)$ is a normal subgroup of G. Any element within the intersection $\bigcap_{n\in\mathbb{N}} \operatorname{Stab}_{\mathsf{G}(\mathcal{A})}(A^n)$ must fix every A^n, so the intersection is the trivial group.

Fix $g, h \in \mathsf{G}(A)$. If $g(\operatorname{Stab}_{\mathsf{G}(\mathcal{A})}(A^n)) = h(\operatorname{Stab}_{\mathsf{G}(\mathcal{A})}(A^n))$ then $h^{-1}g \in \operatorname{Stab}_{\mathsf{G}(\mathcal{A})}(A^n)$ and $h(v) = g(v)$ for all $v \in A^n$. Each coset of $\mathsf{G}(\mathcal{A})/\operatorname{Stab}_{\mathsf{G}(\mathcal{A})}(A^n)$ can therefore be identified with a distinct permutation in S_{A^n}. As S_{A^n} is a finite collection, it follows that the quotient $\mathsf{G}(\mathcal{A})/\operatorname{Stab}_{\mathsf{G}(\mathcal{A})}(A^n)$ is finite, so $\mathsf{G}(\mathcal{A})$ is residually finite. ∎

With this, we reach the following conclusion:

Corollary 5. If G is non-Abelian, then $G \operatorname{wr} \mathbb{Z}$ is not self similar.

3.5 The Group Structure of Cayley Machines

It was determined in Section 3.4.1 that when G is a finite Abelian group, the group generated by its Cayley machine $\mathcal{C}(G)$ is isomorphic to $G \operatorname{wr} \mathbb{Z}$. When G is a finite non-Abelian group, however, L_G is not self-similar, as was determined in Section 3.4.2. It is natural to then ask what the group structure of $\mathsf{G}(\mathcal{C}(G))$ is in this finite non-Abelian case. The results given here are also found in [17].

Definition 30. Let $\mathcal{A} = (Q, A, \delta, \lambda)$ be an automaton. A letter $\alpha \in A$ is said to be a *reset* if there is $p \in Q$ for which $\delta(q, \alpha) = p$ for all $q \in Q$. If every $\alpha \in A$ is a reset (potentially to distinct states of $\mathcal{A}$) then $\mathcal{A}$ is said to be a *reset automaton.*

Example 14. The automaton that generates L_2 is a reset automaton. See Figure 1.2.

Definition 31. Let $\mathcal{A} = (Q, A, \delta, \lambda)$ be a reset automaton. Then $Q|_\alpha$ denotes the common state $q|_\alpha$, where $q \in Q$ and $\alpha \in A$. For every $q \in Q$, the function $Q|_{q(\cdot)}$ is called q's *modified state function.*

Example 15. The states of the automaton in Figure 1.2 are denoted a and b. The modified state function of a is

$$Q|_{a(\,\cdot\,)} = \begin{cases} 0 & : Q|_{a(0)} = Q|_1 = a \\ 1 & : Q|_{a(1)} = Q|_0 = b \end{cases}$$

and the modified state function of b is

$$Q|_{b(\,\cdot\,)} = \begin{cases} 0 & : Q|_{b(0)} = Q|_0 = b \\ 1 & : Q|_{b(1)} = Q|_1 = a \end{cases}.$$

Definition 32. Let $f \in \operatorname{Aut} A^\omega$, where A is a finite set. Then the *depth* of f is the least n for which $f(u^\frown v) = f(u)^\frown v$ whenever $u \in A^n$ and $v \in A^\omega$. If no such n exists, we say that the depth of f is infinite.

Remark 5. If $\mathcal{A} = (Q, A, \delta, \lambda)$ is an invertible reset automaton with $A = \{\alpha_i\}_{1 \le i \le n}$, then $q = \lambda_q(Q|_{\alpha_1}, \ldots, Q|_{\alpha_n})$.

We assume the following:

> (**Theorem 11**) Let $\mathcal{A} = (Q, A, \delta, \lambda)$ be an invertible reset automaton with more than one state and every modified state function distinct. Then $\mathsf{S}(\mathcal{A})$ is a free semigroup on Q.

It is proved in [17]. The proof is also provided in Appendix A. We also recall Lemma 8, which sees use in this section:

> (**Lemma** 8) Let G be a group, let $H \le G$, and let $a \in G$ be such that $G = \langle H, a\rangle$. Then the normal closure $\operatorname{ncl}(H)$ is $\langle {}^{a^m}h : h \in H, m \in \mathbb{Z}\rangle$ and $G = (\operatorname{ncl}(H))\langle a\rangle$.

Lemma 14. Let $\mathcal{A} = (Q, A, \delta, \lambda)$ be an invertible reset automaton. Suppose $A = \{\alpha_i\}_{1 \le i \le n}$, and let $p, q \in Q$ be states of the automaton, not necessarily distinct. If $f \in \operatorname{Aut} A^\omega$ has depth m, then pfq^{-1} has depth at most $m + 1$.

Proof. As $\mathcal{A}$ is a reset automaton, $q = \lambda_q(Q|_{\alpha_1}, \ldots, Q|_{\alpha_n})$ and $p = \lambda_p(Q|_{\alpha_1}, \ldots, Q|_{\alpha_n})$. If the depth of f is 0, then $f = \mathrm{Id}_A(\mathrm{Id}_{A^{\leq\omega}}, \ldots, \mathrm{Id}_{A^{\leq\omega}})$, so

$$
\begin{aligned}
pfq^{-1} &= \lambda_p(Q|_{\alpha_1}, \ldots, Q|_{\alpha_n}) \cdot \mathrm{Id}_A(\mathrm{Id}_{A^{\leq\omega}}, \ldots, \mathrm{Id}_{A^{\leq\omega}}) \cdot \lambda_q^{-1}\Big(Q|^{-1}_{q^{-1}(\alpha_1)}, \ldots, Q|^{-1}_{q^{-1}(\alpha_n)}\Big) \\
&= \lambda_p(Q|_{\alpha_1}, \ldots, Q|_{\alpha_n}) \cdot \lambda_q^{-1}\Big(Q|^{-1}_{q^{-1}(\alpha_1)}, \ldots, Q|^{-1}_{q^{-1}(\alpha_n)}\Big) \\
&= (\lambda_p\lambda_q^{-1})\Big(Q|_{q^{-1}(\alpha_1)}Q|^{-1}_{q^{-1}(\alpha_1)}, \ldots, Q|_{q^{-1}(\alpha_n)}Q|^{-1}_{q^{-1}(\alpha_n)}\Big) \\
&= (\lambda_p\lambda_q^{-1})(\mathrm{Id}_{A^{\leq\omega}}, \ldots, \mathrm{Id}_{A^{\leq\omega}})
\end{aligned}
$$

has depth at most 1. Let $m \in \mathbb{N}$ and suppose pfq^{-1} has depth at most $\ell+1$ whenever f has depth ℓ, for all $\ell < m$. Then if $f = \lambda_f(f|_{\alpha_1}, \ldots f|_{a_n})$ has depth m, each $f|_{\alpha_i}$ has depth at most $m-1$, so each $Q|_{f(\alpha_i)}f|_{q^{-1}(\alpha_i)}Q|^{-1}_{\alpha_i}$ has depth at most m by the induction hypothesis. Thus

$$pfq^{-1} = (\lambda_q\lambda_f\lambda_q^{-1})\big(Q|_{f(\alpha_1)}f|_{q(\alpha_1)}Q|^{-1}_{\alpha_1}, \ldots, Q|_{f(\alpha_n)}f|_{q(\alpha_n)}Q|^{-1}_{\alpha_n}\big)$$

has depth at most $m+1$. ■

This lemma is used to derive

Theorem 9. Let $\mathcal{A} = (Q, A, \delta, \lambda)$ be an invertible reset automaton with $A = \{\alpha_i\}_{1\leq i\leq n}$, and let $p \in Q$ be a state. Then there is a subgroup L of $\mathsf{G}(\mathcal{A})$ such that

1. $L \leq S_A$,
2. $\mathsf{G}(\mathcal{A}) = \langle L, p\rangle = \mathrm{ncl}(L)\langle p\rangle$, and
3. $\mathrm{ncl}(L)$ is locally finite.

Proof. Let $L = \langle pq^{-1} : q \in Q \rangle$. Then as

$$\begin{aligned} pq^{-1} &= \lambda_p(Q|_{\alpha_1}, \ldots, Q|_{\alpha_n}) \cdot \lambda_q^{-1}\Big(Q|^{-1}_{q^{-1}(\alpha_1)}, \ldots, Q|^{-1}_{q^{-1}(\alpha_n)}\Big) \\ &= (\lambda_p \lambda_q^{-1})\Big(Q|_{q^{-1}(\alpha_1)} Q|^{-1}_{q^{-1}(\alpha_1)}, \ldots, Q|_{q^{-1}(\alpha_n)} Q|^{-1}_{q^{-1}(a_n)}\Big) \\ &= \lambda_p \lambda_q^{-1} \end{aligned}$$

is in S_A, it follows that $L \leq S_A$. Also, every state $q \in Q$ is in $\langle L, p\rangle$ by $q = (\lambda_q \lambda_p^{-1})p$, so Lemma 8 gives that $\mathsf{G}(\mathcal{A}) = \operatorname{ncl}(L)\langle p \rangle$.

Fix $s \in \mathbb{N}$ (*not* in $\mathbb{Z}$). As $\ell(\mathrm{Id}_{A^{\leq\omega}}, \ldots, \mathrm{Id}_{A^{\leq\omega}})$ has depth 1 for each $\ell \in L$, every element ${}^{q^s}\ell$ has depth at most $s+1$ by Lemma 14 and induction on s. Thus the subgroup $M = \langle {}^{p^s}\ell : \ell \in L, s \in \mathbb{N}\rangle$ is generated by finite depth elements. But to be invertible of depth s means to imbed in some $\operatorname{Aut} A^s \leq S_{A^s}$, a finite collection, so for any $m_1, \ldots, m_k \in M$ with maximum depth s, it follows $\langle m_1, \ldots, m_k \rangle$ is finite. Suppose $n_1, \ldots, n_r \in \operatorname{ncl}(L)$ by $n_i = {}^{p^{s_{i1}}}\ell_{i1} \cdots {}^{p^{s_{it_i}}}\ell_{it_i}$, where $s_{i1}, \ldots, s_{it_i} \in \mathbb{Z}$. If we choose $s_i \in \mathbb{Z}$ so that $s_i \geq -\min\{s_{i1}, \ldots, s_{it_i}\}$ holds, then

$${}^{p^{s_i}}n_i = p^{s_i}({}^{p^{s_{i1}}}\ell_{i1} \cdots {}^{p^{s_{it_i}}}\ell_{it_i})p^{-s_i} = {}^{p^{s_{i1}+s_i}}\ell_{i1} \cdots {}^{p^{s_{it_i}+s_i}}\ell_{it_i}.$$

In particular, the conjugate is in M. Letting $s = \max\{s_1, \ldots, s_k\}$, it follows ${}^{p^s}\langle n_1, \ldots, n_k\rangle$ is a subgroup of M. As $\langle n_1, \ldots, n_k \rangle$ is isomorphic to ${}^{p^s}\langle n_1, \ldots, n_k\rangle$, it is finite. Thus $\operatorname{ncl}(G)$ is locally finite. ■

Corollary 6. If $\mathcal{A} = (Q, A, \delta, \lambda)$ is an invertible reset automaton and some $q \in Q$ is infinite order, then there is $G \leq \mathsf{G}(\mathcal{A})$ for which $\mathsf{G}(\mathcal{A}) = \operatorname{ncl}(G) \rtimes \langle q \rangle$.

Proof. As $\operatorname{ncl} G$ is locally finite and every q^k is infinite, $\operatorname{ncl}(G) \cap \langle q \rangle = \{1\}$. Thus $\mathsf{G}(\mathcal{A}) = \operatorname{ncl}(G)\langle q \rangle = \operatorname{ncl}(G) \rtimes \langle q \rangle$ by Lemma 1. ■

As in Section 3.4.1, because the elements of G are already members of a group, $\hat{g}$ is used refer to g as an element of $\operatorname{Aut} G^{\leq\omega}$.

Theorem 10. Let G be a non-trivial finite group

1. $\mathcal{C}(G)^{-1}$, the inverse automaton of the Cayley machine of G, is a reduced reset automaton with the resets going to distinct states.

2. $\mathsf{S}(\mathcal{C}(G))$, the semigroup generated by the Cayley machine of G, is a free semigroup on $\{\hat{g} : g \in G\}$.

3. $\mathsf{G}(\mathcal{C}(G))$, the group generated by the Cayley machine of G, equals $N \rtimes \mathbb{Z}$ where N is locally finite.

Proof. By Definition 11, we have that $\hat{g}^{-1}|_h = \hat{h}^{-1}$ and $\hat{g}^{-1}(h) = gh$ for each $g, h \in G$. It follows that $\mathcal{C}(G)^{-1}$ is reset, as on input h, every state of $\mathcal{C}(G)^{-1}$ goes to state $\hat{h}^{-1}$. It is also reduced, as $\hat{g}^{-1}(h) = gh \neq gk = \hat{g}^{-1}(k)$ unless $h = k$. This proves Item 1

Theorem 11 gives that $\mathsf{S}(\mathcal{C}(G)^{-1})$ is free on $\{\hat{g}^{-1} : g \in G\}$, so then $\mathsf{S}(\mathcal{C}(G))$ is free on $\{\hat{g} : g \in G\}$—otherwise we could carry the relations on the $\hat{g}$'s over to their inverses. As $\mathsf{S}(\mathcal{C}(G)^{-1})$ is free on $\{\hat{g}^{-1} : g \in G\}$, it follows each $\hat{g}^{-1}$ is infinite order. This proves Item 2

By Item 1 and Corollary 6, we have that

$$\mathsf{G}(\mathcal{C}(G)^{-1}) = \mathrm{ncl}\Big\langle \hat{g}^{-1}(\hat{h}^{-1})^{-1} : \hat{g}^{-1}, \hat{h}^{-1} \in G^{-1} \Big\rangle \rtimes \mathbb{Z} = \Big\langle \hat{g}^{-1}\hat{h} : g, h \in G \Big\rangle \rtimes \mathbb{Z}.$$

Local finiteness follows from Theorem 9. As $\mathsf{G}(\mathcal{C}(G)) = \mathsf{G}(\mathcal{C}(G)^{-1})$, this shows Item 3. ■

Lastly, it is worth mentioning the paper [19] by Ning Yang. Here, the author finds the presentation of $\mathsf{G}(\mathcal{C}(G))$ when G is 2-step nilpotent with all squares central. They further demonstrate that $\mathsf{G}(\mathcal{C}(G))$ belongs, in this case, to a class of groups known as *cross-wired lamplighters*. Cross-wired lamplighter groups are brought about by looking at certain subgroups of isometries of Diestel-Leader graphs [4, 18, 5, 12].

Appendix A: A Theorem of Silva and Steinberg

In Section 3.5, it was stated that Theorem 10 relied on the following result:

> (**Theorem** 11) If $\mathcal{A} = (Q, A, \delta, \lambda)$ is an invertible reset automaton with more than state and with distinct modified state functions, then $\mathsf{S}(\mathcal{A})$ is a free semigroup on $Q \subseteq \operatorname{Aut} A^{\leq \omega}$.

Thus when G is a finite Abelian group, the lamplighter group L_G has non-trivial relations and nevertheless contains a free semigroup. This result is presented here. It and Theorem 6 are originally from [17], which includes along with these two theorems a wealth of discussion and results that build atop them.

Lemma 15. Let $\mathcal{A} = (Q, A, \delta, \lambda)$ be a reset automaton. Then

$$q(\alpha_1 \cdots \alpha_n) = q(\alpha_1)^\frown Q|_{\alpha_1}(\alpha_2)^\frown \cdots ^\frown Q|_{\alpha_{n-1}}(\alpha_n).$$

In particular, if we denote $q(\alpha_1 \cdots \alpha_n) = \beta_1 \cdots \beta_n$, then each β_i only depends on α_{i-1} and α_i, unless $i = 1$, in which case β_1 depends solely on q.

Proof. If $n = 1$ then this is immediate, and if $n = 2$ then $q|_{\alpha_1} = Q|_{\alpha_1}$ as $\mathcal{A}$ is reset. If it holds for all $m \leq n$, then as $q|_{\alpha_1 \cdots \alpha_n} = (q|_{\alpha_1 \cdots \alpha_{n-1}})|_{\alpha_n} = Q|_{\alpha_n}$, it follows that

$$q(\alpha_1 \cdots \alpha_{n+1}) = q(\alpha_1 \cdots \alpha_n)^\frown q|_{\alpha_1 \cdots \alpha_n}(\alpha_{n+1}) = q(\alpha_1)^\frown Q|_{\alpha_1}(\alpha_2)^\frown \cdots ^\frown Q|_{\alpha_n}(\alpha_{n+1}).$$

∎

The result can be generalized:

Lemma 16. Let $\mathcal{A} = (A, Q, \delta, \lambda)$ be a reset automaton, let $q^{(1)}, \ldots, q^{(m)} \in Q$, and let $\alpha_1, \ldots, \alpha_n \in A$. If $(q^{(m)} \cdots q^{(1)})(\alpha_1 \cdots \alpha_n) = \beta_1, \cdots \beta_n$ and $i > m$, then β_i depends solely on $\alpha_{i-m}, \cdots, \alpha_i$. Moreover, $\alpha_i \mapsto \beta_i$ is a permutation.

Proof. Lemma 15 and its commentary give the base case for any n and i. If we assume that the statement holds up to some $m \in \mathbb{N}_+$, then if

$$(q^{(m+1)}q^{(m)} \cdots q^{(1)})(\alpha_1 \cdots \alpha_n) = q^{(m+1)}(\beta_1 \cdots \beta_n) = \gamma_1 \cdots \gamma_n$$

and $i > m+1 > 1$, it follows from the base case that γ_i depends solely on β_i and β_{i-1} and that the function $\beta_i \mapsto \gamma_i$ is a permutation. From the inductive case, β_i and β_{i-1} depend respectively on $\alpha_{i-m}, \dots, \alpha_i$ and $\alpha_{i-(m+1)}, \dots, \alpha_{i-1}$, and $\alpha_i \mapsto \beta_i$ is a permutation. Thus γ_i depends on $\alpha_{i-(m+1)}, \dots, \alpha_i$, and $\alpha_i \mapsto \gamma_i$ is a permutation. ■

Lemma 17. Let $|X| \geq 2$ and let $\equiv$ be a congruence on $X^{<\omega}$ such that if $u \equiv v$ and $|u| = |v|$, then $u = v$. Then $\equiv$ is the trivial congruence, meaning $u \equiv v$ if and only if $u = v$ for every $u, v \in A^{<\omega}$.

Proof. Suppose $\equiv$ is not the trivial congruence and let $u \equiv v$ despite $u \neq v$. Let w be the largest common prefix of u and v. That is, $w = \max\{\tilde{w} \in X^{<\omega} : \tilde{w} \leq u, \tilde{w} \leq v\}$, where the maximum is with respect to $A^{<\omega}$'s ordering (see Definition 8). Let $x, y \in A^{<\omega}$ be such that $u = w^\frown x$ and $v = w^\frown y$. Then neither $x \not\leq y$ nor $y \not\leq x$, and in fact if $x = \xi_1 \cdots \xi_n$ and $y = \eta_1 \cdots \eta_m$, then $\xi_1 \neq \eta_1$, so $x^\frown y \neq y^\frown x$. But $w \equiv w$, so as $\equiv$ is a congruence it follows $x \equiv y$, or else $u \not\equiv v$. Thus $x^\frown y \equiv y^\frown x$. Therefore, if $\equiv$ is not the trivial congruence, then there are congruent words of equal length that are nonetheless distinct. ■

Theorem 11. If $\mathcal{A} = (Q, A, \delta, \lambda)$ is an invertible reset automaton with more than state and with distinct modified state functions, then $\mathsf{S}(\mathcal{A})$ is a free semigroup on $Q \subseteq \operatorname{Aut} A^{\leq\omega}$.

Proof. For any distinct $p, q \in Q$, since $Q|_{p(\,\cdot\,)} \neq Q|_{q(\,\cdot\,)}$ there is $\alpha \in A$ such that $Q|_{p(\alpha)} \neq Q|_{q(\alpha)}$. Thus $p(\alpha) \neq q(\alpha)$, so the states are distinct as functions on $\operatorname{Aut} A^{\leq\omega}$ (in fact they are distinct as functions on S_A). The automaton $\mathcal{A}$ is therefore reduced, and it makes sense to view Q as a subset of $\operatorname{Aut} A^{\leq\omega}$.

Let $\equiv$ be the congruence relation on $Q^{<\omega}$ given by $u \equiv v$ if and only if u and v are equal as functions in $\operatorname{Aut} A^{<\omega}$. Suppose for contradiction that $\mathsf{S}(\mathcal{A})$ is not free on Q—that is, $\equiv$ is not the trivial congruence relation. Then by Lemma 17 there are $p^{(1)}, \ldots, p^{(n)} \in Q$ and $q^{(1)}, \ldots, q^{(n)} \in Q$ such that $p^{(n)} \cdots p^{(1)}$ and $q^{(n)} \cdots q^{(1)}$ are distinct as words in $Q^{<\omega}$, yet $p^{(n)} \cdots p^{(1)} \equiv q^{(n)} \cdots q^{(1)}$. In particular, $p^{(i)} \neq q^{(i)}$ for some i. As $\mathsf{S}(\mathcal{A})$ imbeds in $\mathsf{G}(A)$, it follows that $\mathsf{S}(\mathcal{A})$ is cancellative, so without loss of generality we may assume $p^{(1)} \neq q^{(1)}$. Let $\alpha_1, \ldots, \alpha_n \in A$; denote $(p^{(i)} \cdots p^{(1)})(\alpha_1 \cdots \alpha_n) = \beta_{i+1,1} \cdots \beta_{i+1,n}$ and $(q^{(i)} \cdots q^{(1)})(\alpha_1 \cdots \alpha_n) = \gamma_{i+1,1} \cdots \gamma_{i+1,n}$. We utilize the following lemma:

Lemma 18. For all $i \geq 1$, one can choose $\alpha_1 \cdots \alpha_i$ such that $Q|_{\beta_{i+1,i}} \neq Q|_{\gamma_{i+1,i}}$.

Proof. Since the modified state functions of $\mathcal{A}$ are all distinct, it follows $\beta_{21} = p^{(1)}(\alpha_1) \neq q^{(1)}(\alpha_1) = \gamma_{21}$ for some $\alpha_1 \in A$, so $Q|_{\beta_{21}} = Q|_{p^{(1)}(\alpha_1)} \neq Q|_{q^{(1)}(\alpha_1)} = Q_{\gamma_{21}}$. Suppose that $\alpha_1, \ldots, \alpha_i$ have been chosen such that $Q|_{\beta_{j+1,j}} \neq Q|_{\gamma_{j+1,j}}$ for every $j \leq i$. Then as the modified state functions of $\mathcal{A}$ are all distinct, there is $\alpha \in A$ for which $Q|_{(Q|_{\beta_{i+1,i}}(\alpha))} \neq Q|_{(Q|_{\gamma_{i+1,i}}(\alpha))}$. Note that $\beta_{i+2,i+1} = Q|_{\beta_{i+1,i}}(\beta_{i+1,i+1})$ and $\gamma_{i+2,i+1} = Q|_{\gamma_{i+1,i}}(\gamma_{i+1,i+1})$ by Lemma 15, and by Lemma 16 $\beta_{i+1,i+1} = \gamma_{i+1,i+1}$ (as these are from applying $p^{(i)} \cdots p^{(1)}$ and $q^{(i)} \cdots q^{(1)}$, respectively). The map $\alpha_{i+1} \mapsto \beta_{i+1,i+1}$ is a permutation by Lemma 16 as well. In particular, α_{i+1} can be chosen such that $\beta_{i+1,i+1} = \alpha = \gamma_{i+1,i+1}$. Thus $Q|_{\beta_{i+2,i+1}} = Q|_{Q|_{\beta_{i+1,i}}(\beta_{i+1,i+1})} \neq Q|_{Q|_{\gamma_{i+1,i}}(\gamma_{i+1,i+1})} = Q|_{\gamma_{i+2,i+1}}$ ∎

Since $Q|_{\beta_{n+1,n}} \neq Q|_{\gamma_{n+1,n}}$ and $\mathcal{A}$ is a reset automaton, it follows that $\beta_{n+1,n} \neq \gamma_{n+1,n}$, so $p^{(n)} \cdots p^{(1)}(\alpha_1 \cdots \alpha_n) \neq q^{(n)} \cdots q^{(1)}(\alpha_1 \cdots \alpha_n)$. Thus $p^{(n)} \cdots p^{(1)}$ and $q^{(n)} \cdots q^{(1)}$ are distinct functions, contradicting their equivalence under the congruence. ∎

Corollary 7. If $\mathcal{A} = (Q, A, \delta, \lambda)$ is a reduced reset automaton with its resets to distinct states and $|A| \geq 1$, then $\mathsf{S}(\mathcal{A})$ is a free semigroup on $Q \subseteq \operatorname{Aut} A^{\leq\omega}$.

Every λ_q is invertible, or else there are $\alpha, \beta \in A$ with $\lambda_q(\alpha) = \lambda_q(\beta)$. As every state is distinct ($\mathcal{A}$ is reduced) and $\mathcal{A}$ is reset, it follows every λ_q—so every modified state function—is distinct. ∎

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