# Geometry of Nearest Integer Continued Fractions

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## Introduction

What is a SCF (Simple Continued Fraction)?

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = \underbrace{[a_0; a_1, a_2, \dots]}_{\text{Quotients}} \quad \text{where } a_i \in \mathbb{Z}^+$$

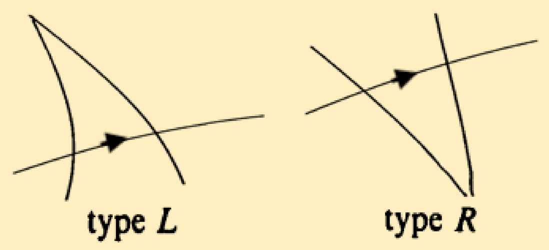
What is a NICF (Nearest Integer Continued Fraction) ?

$$a_0 + \cfrac{e_1}{a_1 + \cfrac{e_2}{\ddots}} = \underbrace{[a_0; (e_1/a_1), (e_2/a_2)\dots]}_{\text{Quotients permitting negative signs}} \quad \text{where } e_i \in \{\pm 1\}$$

Whereas the SCF has strictly positive quotients $a_i \in \mathbb{Z}^+$, the NICF permits negative signs and its quotients converge faster than its SCF counterpart.

## Cutting Sequence on the Hyperbolic Plane

As we cut across Farey triangles on the hyperbolic plane $\mathbb{H}$, each defined by the vertices $(\frac{a}{b}, \frac{a+c}{b+d}, \frac{c}{d})$, we label each triangle either $R$ or $L$ depending on the orientation where two sides of the triangle join at a vertex.




## Theorem (Series's Theorem A)

*For an oriented geodesic $\gamma$ with endpoints $\gamma_+, \gamma_-$:*

- *When $\gamma_+ = [n_1; n_2, \dots]$ and $\gamma_- = -[0; n_0, n_{-1}, \dots]$, the geodesic has cutting sequence $\dots L^{n_{-1}} R^{n_0} x L^{n_1} R^{n_2} \dots$.*
- *When $\gamma_+ = -[n_1; n_2, \dots]$ and $\gamma_- = [0; n_0, n_{-1}, \dots]$, the geodesic has cutting sequence $\dots R^{n_{-1}} L^{n_0} x R^{n_1} L^{n_2} \dots$.*

*where $x$ marks where the geodesic crosses the imaginary axis.*

## Example

Simple Continued Fraction:

$$\frac{8}{3} = \frac{2(3)+2}{3} = 2 + \frac{2}{3} = 2 + \cfrac{1}{\cfrac{3}{2}} = 2 + \cfrac{1}{1 + \cfrac{1}{2}}$$

Nearest Integer Continued Fraction:

$$\frac{8}{3} = \frac{3(3)-1}{3} = 3 - \frac{1}{3}$$

## Geometric Analogue: the Farey Tesselation (Code developed by us)

Neighbors $\frac{a}{b}$ and $\frac{c}{d}$, where $\frac{a}{b} > \frac{c}{d}$, are connected by a geodesic if and only if $ad - bc = 1$.

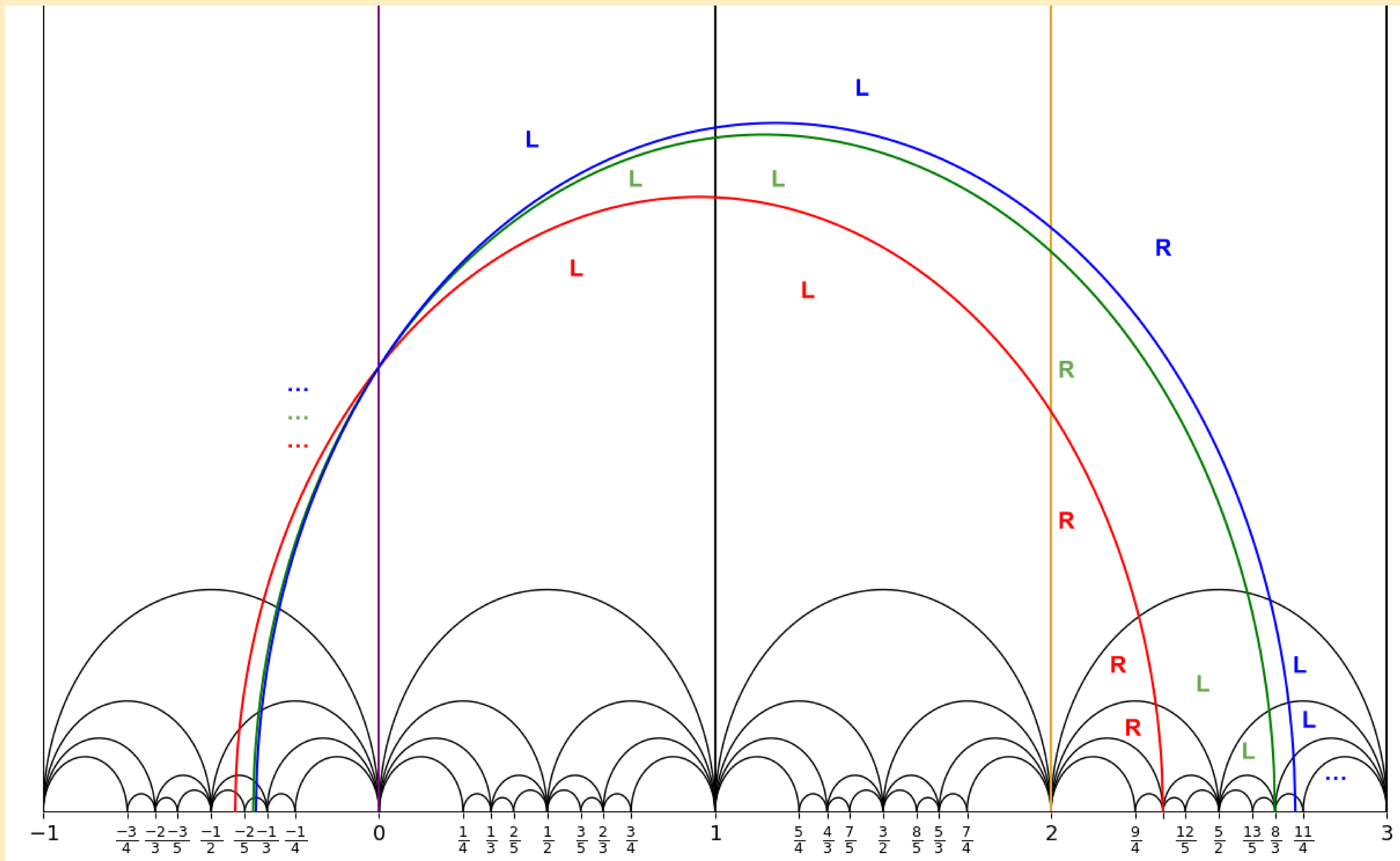



Figure:
Cutting Sequence for green (middle) geodesic... $xL^2R^1L^3$
Cutting Sequence for red (bottom) geodesic ... $xL^2R^3$
Cutting Sequence for blue (top) geodesic ... $xL^2RL^2\dots$

## Farey Continued Fraction and Our Proposed Nearest Gauss Map

The Farey Gauss map is:

$$F(x) := \begin{cases} \frac{1}{x} - 1, & \text{if } x \in (0, \infty) \\ -\frac{1}{x} - 2, & \text{if } x \in [-1, 0, ) \end{cases} \quad \text{where } (f_i/b_i) = \begin{cases} (+1/1) & \text{if } F^i(x) \in (0, \infty) \\ (-1/2) & \text{if } F^i(x) \in [-1, 0) \end{cases}$$

$$\text{A fraction } x = n_0 + \cfrac{1}{n_{-1} + \cfrac{1}{\ddots}} \quad \text{becomes} \quad (1/1)(-1/2)^{n_0}(1/1)(-1/2)^{n_{-1}-1}\dots$$

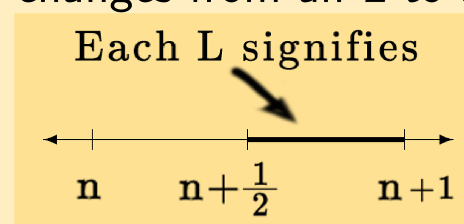
Similarly, we observed that our NICF algorithm follows a comparable pattern, albeit where we now have to keep track of an additional array $\mathcal{K} = [k_1, k_2, \dots]$. The Nearest Gauss Map is then:

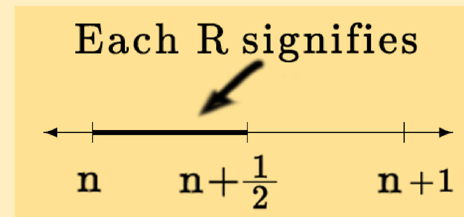
$$T(x) := \begin{cases} \frac{1}{x} - k_i, & \text{if } x \in \left[\frac{1}{k_i + \frac{1}{2}}, \frac{1}{k_i}\right] \\ -\frac{1}{x} + k_i, & \text{if } x \in \left(\frac{1}{k_i+1}, \frac{1}{k_i + \frac{1}{2}}\right) \end{cases} \quad \text{where } (b_i, f_{i+1}) = \begin{cases} (k_i, +1) \text{ if } x \in \left[\frac{1}{k_i+\frac{1}{2}}, \frac{1}{k_i}\right] \\ (k_i, -1) \text{ if } x \in \left(\frac{1}{k_i+1}, \frac{1}{k_i+\frac{1}{2}}\right) \end{cases}$$

$$\text{Example: } \frac{5}{7} = \cfrac{1}{1 + \cfrac{1}{2 + \frac{1}{2}}} \quad \text{becomes} \quad (1/1)(-1/3)(1/2) = 1 - \cfrac{1}{3 + \frac{1}{2}}.$$

## Working Hypothesis

For a cutting sequence $\dots xL^2yL^{n_1-2}R^{n_2}L^{n_3}R^{n_4}L^{n_5}\dots$ where $y$ is where $\gamma$ crosses $x = 2$; rounding occurs in Euclid's Algorithm each time the cutting sequence changes from an $L$ to an $R$.




($R$ $L$) means rounding occurs
$R^2$ means no rounding occurs, continue until $L$.

## Example

Cutting Sequence $xL^2yRL^2RL^2$

Simple Continued Fraction:

$$\frac{30}{11} = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}}$$

Nearest Integer Continued Fraction:

$$\frac{30}{11} = 3 - \frac{3}{11} = 3 - \cfrac{1}{\cfrac{11}{3}} = 3 - \cfrac{1}{4 - \cfrac{1}{3}}$$

## Future Work & Summary

Caroline Series described an explicit geometrical relationship between the cutting sequence of a geodesic and the simple continued fractions. Our continued goal is to show how the cutting sequence acting on the Farey triangles in the hyperbolic plane $\mathbb{H}$ can be mapped to a new algorithm on the same fundamental tile $(0, 1, \infty)$ or some shift of it, which corresponds to the Nearest Integer Continued Fractions. This would mark a first in the literature surrounding Nearest Integer CFs, and provide an explicit relationship between the cutting sequence of geodesics and the Nearest Integer algorithm.

## References

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