

INTRODUCTION

- Network Synchronization
 - Many systems (power grids, traffic, neurons) need their components to move together (synchronize).
 - We model this using the **Kuramoto Model** to assess a **network of nodes**.
 - Whether the system synchronizes depends on the **coupling strength**.
- The Problem
 - Adding a new connection (an edge) should help a network synchronize, but sometimes it makes things worse.
 - The network then **needs stronger coupling** to stay aligned.
 - This counter-intuitive failure is **Braess's Paradox**.

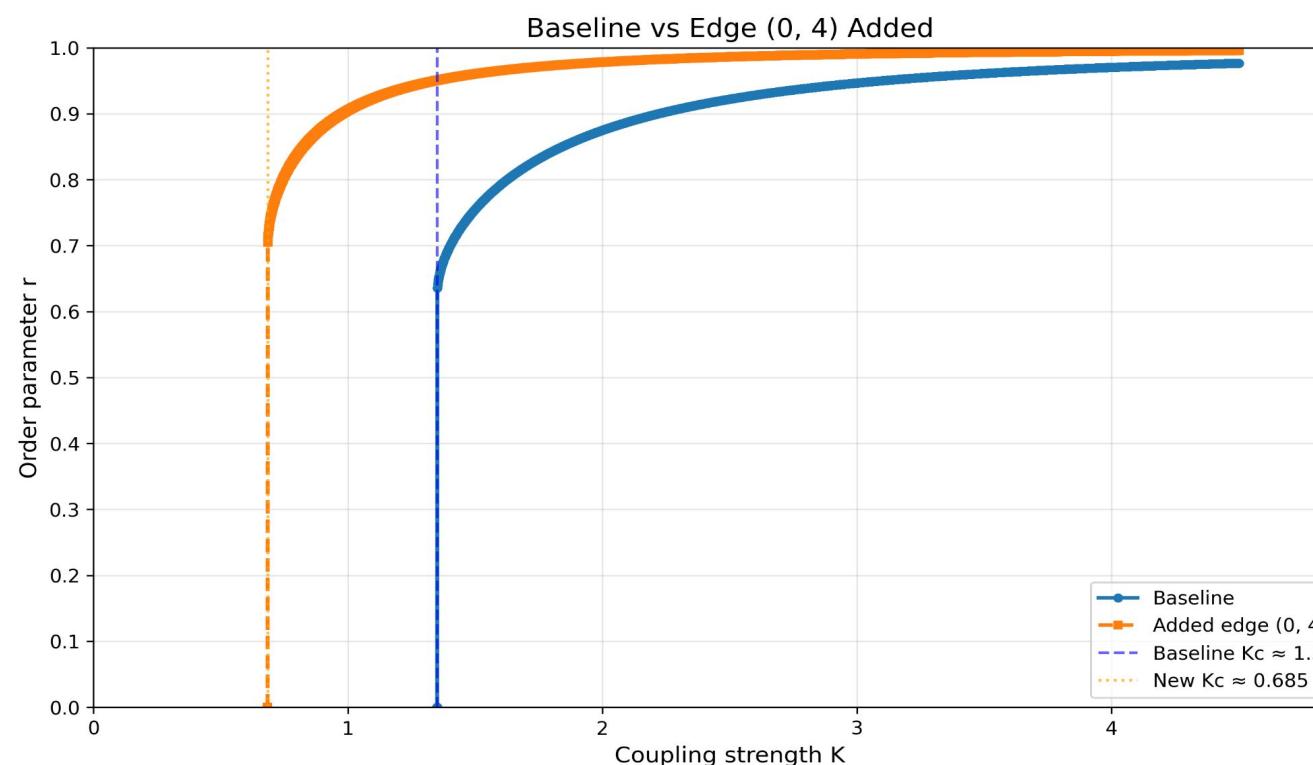
Our Goal: Can we develop an analytical expression that predicts which added edges will cause Braess's paradox in oscillator networks?

The Kuramoto Model

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^N W_{ij} \sin(\theta_j - \theta_i)$$

- θ_i → The phase of oscillator *i*
- W_{ij} → Adjacency matrix between *i* and *j*
- ω_i → Natural frequency of *i*
- $\sin(\theta_j - \theta_i)$ → Interaction between *i* and *j* pulling phases
- K → Coupling strength constant
- $\sum_j W_{ij} \sin(\theta_j - \theta_i)$ → Total influence of all of *i*'s neighboring nodes

Braess Paradox



This Graph Depicts the change in Coupling Strength and the jump in *K_c* value that could occur.

METHODOLOGY

Implicit Function Theorem (IFT)

- IFT tells us how the synchronized solution changes when we change the network.
 - Like adding an edge.
- It requires a full rank Jacobian

Projection & Cokernal

$$DG = \begin{bmatrix} D_\theta F & 0 & F_K & F_a \\ D_\theta(DFv) & DF & (DF)_K v & (DF)_a v \\ \vdots & \text{constraints} & \vdots & \vdots \end{bmatrix}$$

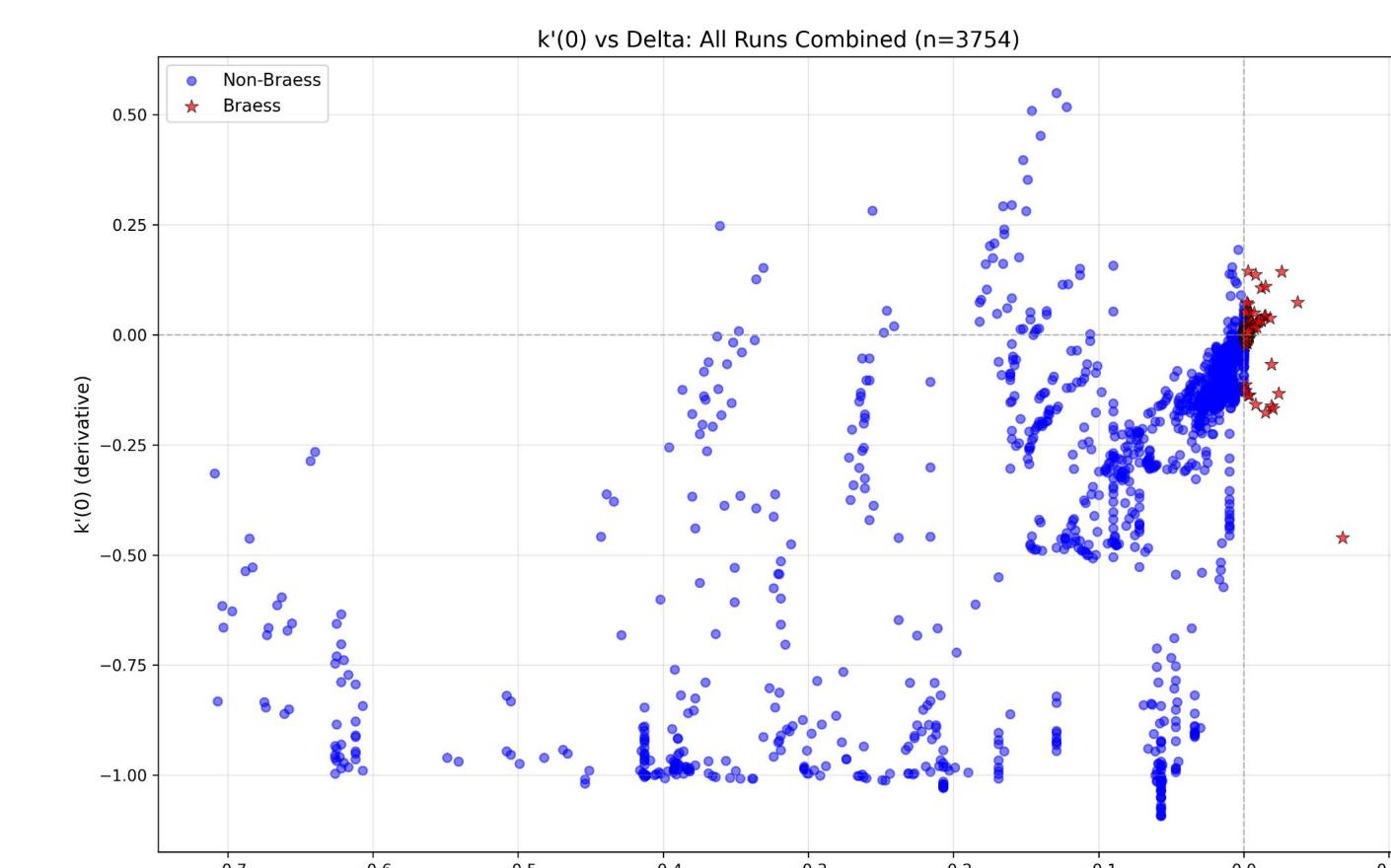
Cokernel vectors: $\psi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\psi_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \psi_i^T DG = 0$.

- The cokernel of DG is two-dimensional, spanned by ψ_1 and ψ_2 .
- They represent the directions where the system loses rank at the critical coupling.
- Projection onto ψ_1 and ψ_2 removes those unsolvable direction.
- This restores the right dimensions so we can apply IFT.

Augmented System (G)

$$G(\theta, v, K_c, a) = \begin{bmatrix} F(\theta, K_c) \\ Jv \\ 1^T v \\ \theta_1 \\ v^T v - 1 \end{bmatrix} = 0.$$

- Each row of G enforces a condition:
 - $F(\theta, K_c) = 0 \rightarrow$ Steady state equation (when our system is in sync).
 - $Jv = 0 \rightarrow$ Critical point where our system loses stability (eigenvalue of $v = 0$).
 - $1^T v = 0 \rightarrow$ Orthogonality for v to the trivial direction.
 - $\theta_1 = 0 \rightarrow$ Gauge fix for rotational symmetry (used as a reference point).
 - $v^T v - 1 = 0 \rightarrow$ Normalizes eigenvector v to remove scaling freedom.
- Our system, however, does not satisfy the dimensions required for IFT ($R^{(2n+2)} \rightarrow R^{(2n+3)}$).



Example K_c Shift Graph over 10,000+ Edges

Confusion Matrix	True Braess Edge	True Non-Braess Edge
Predicted Braess	500	26
Predicted Non Braess	1300	6432

Metric	Overall Accuracy
Our Analytical Method	80%
Previous	N/A

Next Steps & Discussion

Next Steps

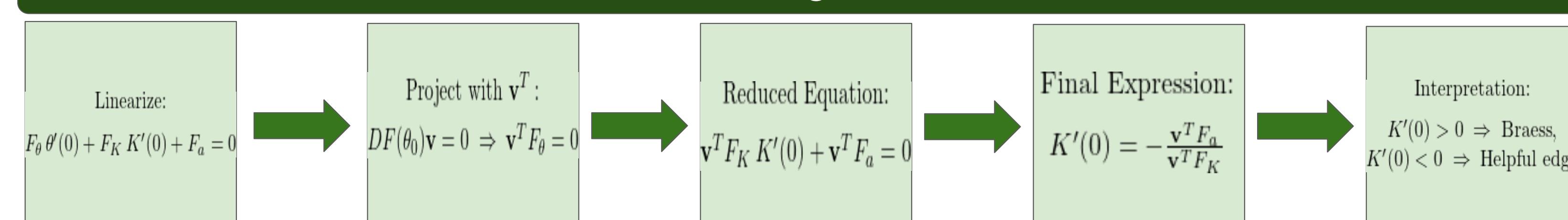
- Validate analytical method on broader network classes
- Test on larger and more realistic networks
- Investigate high false-positive rate

Future Work

- Enhance explainability of predictions
- Identify structural causes behind false positives
- Develop theory linking network structure to Braess behavior?

Citations

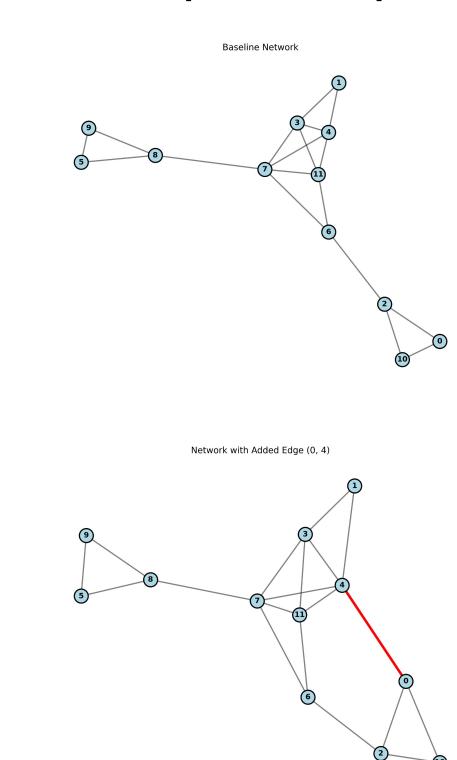
<https://iopscience.iop.org/article/10.1088/1367-2630/14/8/083036>
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Deriving $K'(0)$ 

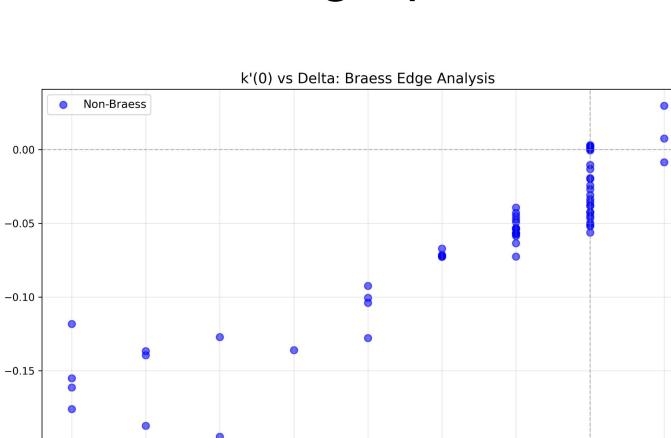
Simulation

Code Logic

Example Graphs



Ideal graph



Our Simulation: 100+ graphs, 10,000+ data points, mix of ER & RGN Graphs

Start

