Geometry of Special Relativity: Twin Paradox

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Introduction

In special relativity, inertial observers in relative motion measure different time intervals and lengths because (i) light has the same speed c in all inertial frames and (ii) physical laws are frame independent.

Definition (Time Dilation)

$$\Delta t = \gamma \Delta \tau$$

- $\Delta\tau$: proper time, measured by a clock at rest with the events.
- Δt : time measured in a frame where that clock moves at speed v.

Moving clocks run slow by a factor γ.

Derivation

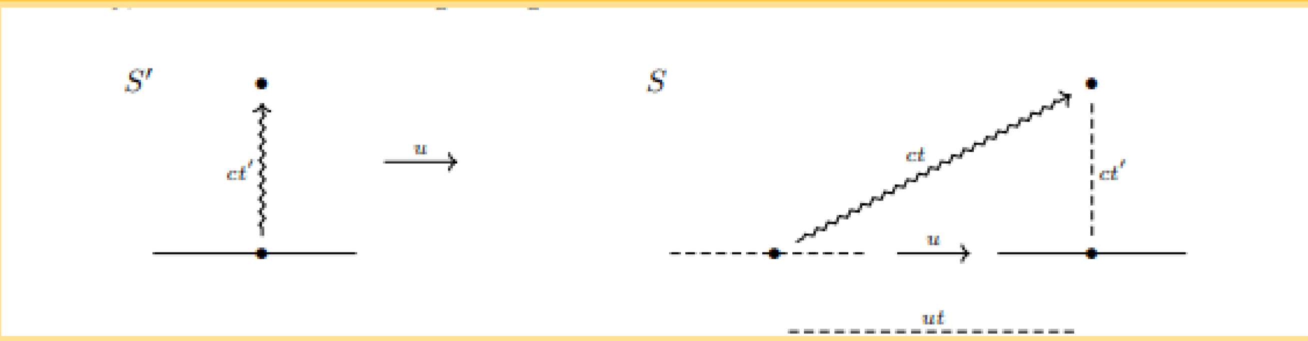


Figure: Light Clock

Rest frame S' : the light bounces straight up and down between mirrors a distance d apart, so one tick takes

$$\Delta\tau = \frac{2d}{c}$$

Lab frame S : the clock moves at speed v, so the light follows a diagonal path. For half a tick, the hypotenuse is $c\Delta t/2$, with vertical leg d and horizontal leg $v\Delta t/2$:

$$(\frac{c\Delta t}{2})^2 = d^2 + (\frac{v\Delta t}{2})^2$$

Solving for Δt gives

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}} = \gamma\Delta\tau$$

Definition (Lorentz Factor γ)

$$\gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

γ tells you exactly how much time intervals dilate and lengths contract at speed v. If $v << c$, then $\gamma \approx 1$ and classical physics holds true.

Lorentz Matrix

Lorentz transformation can be represented as a change-of-basis matrix. We can also always first apply a rotation so that the relative velocity between two intertial observers is entirely in one direction. This allows us to simplify the Lorentz Transform to

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

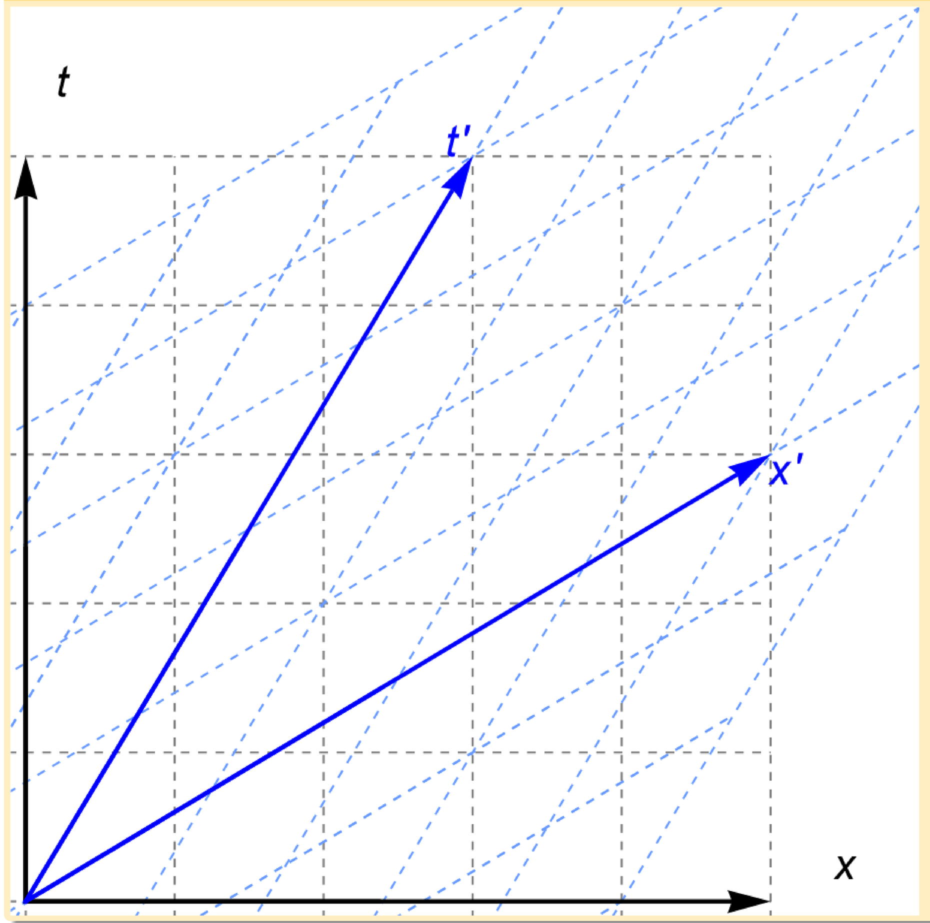
Definition (Space-Time Interval)

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

The spacetime interval Δs^2 measures the interval between two events, and is invariant under the Lorentz Transform. So observers in different inertial reference frames will find two events are separated by the same spacetime interval, despite the observers having different measurements.

- $\Delta s^2 > 0 \Rightarrow$ Timelike interval. Events are causally connected. One event can impact the other.
- $\Delta s^2 = 0 \Rightarrow$ Lightlike interval. Only light could be present at both events.
- $\Delta s^2 < 0 \Rightarrow$ Spacelike interval. Causally disconnected. It is impossible for anything, even light, to go from one event and reach the other.

Change of Basis in Minkowski Space



Application

Space-Time Path

To illustrate how these concepts are used to determine time dilation, consider the following path in space time. This path achieves a maximum distance from the initial event of 1 light year and is periodic on the interval $t \in [0, 5]$. Furthermore, it represents a straight path between the initial event and maximum distance or an elliptical orbit between the two.

$$r(t) = \frac{1}{2}\left[1 + \sin\left(\frac{2\pi t}{5} - \frac{\pi}{2}\right)\right]$$

Time Dilation

First, we find the velocity of this path to use in the Lorentz factor.

$$v(t) = r'(t) = \frac{\pi}{5}\cos\left(\frac{2\pi t}{5} - \frac{\pi}{2}\right)$$

Then, we use the fact that $\Delta t = \gamma\Delta\tau$ to find the total time dilation for the entire path.

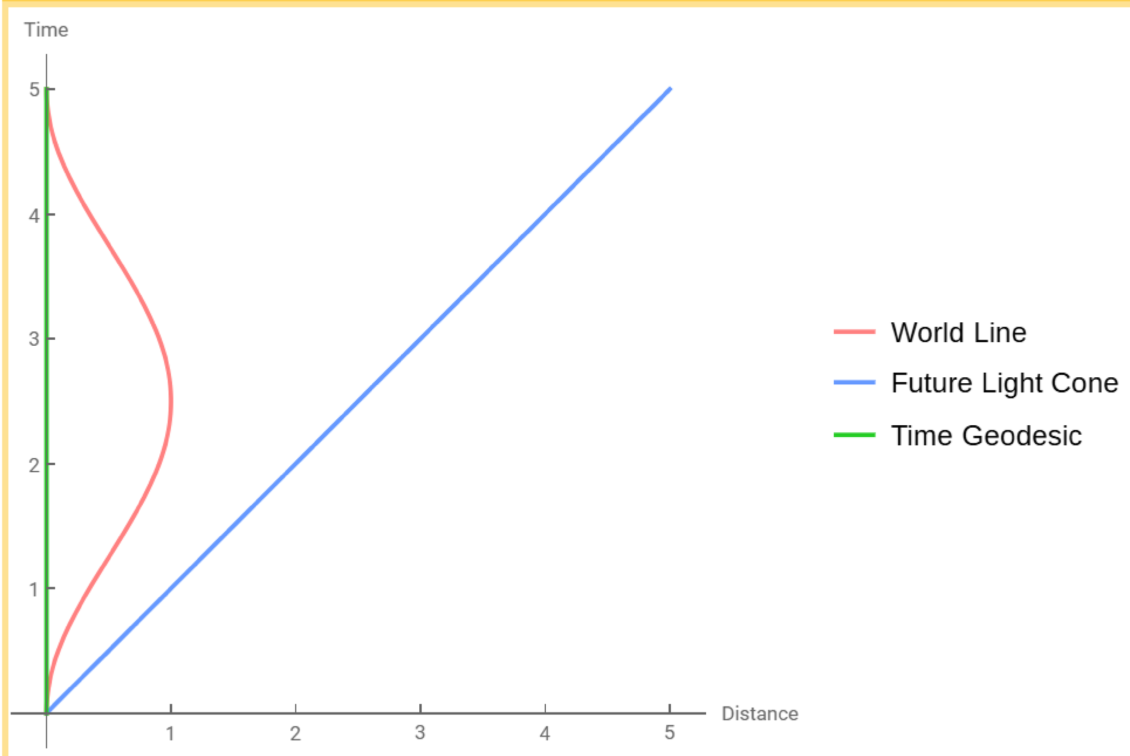
$$\frac{\Delta t}{\gamma} = \Delta\tau \implies d\tau = \frac{dt}{\gamma}$$

Note that $\Delta\tau$ is the proper time of the path and Δt is the proper time of an inertial reference frame.

$$\tau = \int_0^5 \sqrt{1 - \frac{v^2}{c^2}}\, dt = 4.46224$$

Thus, we find that the path experienced approximately 196 fewer days than the inertial reference frame as a result of time dilation.

Path of $r(t)$ in Minkowski Space



Framing the Twin Paradox

Shortly after Einstein published his paper on Special Relativity, Paul Langevin proposed a thought experiment as follows. Suppose there are two twins; one traveling at relativistic speeds through space and one on Earth. When they meet again, the twin traveling at relativistic speeds is younger than the twin on Earth. How can the difference in the time experienced by each twin be reconciled?

Conclusion

The key to explaining this discrepancy is to notice that each twin is in a separate reference frame. We are given that one twin is traveling at near the speed of light, while one is in an inertial or near inertial reference frame on Earth. As such, each twin has a separate path through space-time following the initial event where both twins are on Earth. Thus, the twin traveling at relativistic speeds is moving away from the initial spatial coordinate more quickly than their counterpart, which leads to time dilation in accordance with the Lorentz Factor. Therefore... there is no paradox! Instead, this represents a non-intuitive result produced by special relativity: as an object's velocity approaches the speed of light, the time that object experiences approaches zero. This means that how an object experiences time is related to how it interacts with space.

Acknowledgments

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References

1. Simonetti, J. (1997, October 21). Frequently Asked Questions About Special Relativity — The Twin Paradox https://www1.phys.vt.edu/ jhs/faq/twins.html
2. Rindler, W. (2006). Relativity: Special, General and Cosmological [Review of Relativity: Special, General and Cosmological]. Oxford University Press.
3. Einstein, A. (1922). On the Electrodynamics of Moving Bodies (J. Walker, Ed.) [Review of On the Electrodynamics of Moving Bodies]. (Original work published 1905)