Visualizing Holey Hyperbolic Polyforms

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Project Goals

- Explore useful ways to represent large hyperbolic polyforms both visually and algebraically
- Create examples of large hyperbolic polyforms
- Determine properties of holey polyforms, especially those with the mimimum number of tiles for their number of holes

Definitions

$\{p, q\}$ Tessellation

1. Tessellation - a covering of the plane with tiles
2. p - the number of sides each tile has
3. q - the number of tiles meeting at each vertex

Our project deals only with hyperbolic tessellations, which satisfy the inequality $(p-2)(q-2) > 4$

Polyform

A figure constructed out of a collection of joined tiles from the tessellation.

Hole

In simple terms, a hole is part of the tessellation that's completely surrounded by polyform. More formally, a hole is a bounded component of the complement of the polyform in the tessellation.

The Function $g_{\{p,q\}}(h)$

$g_{\{p,q\}} : \mathbb{N} \to \mathbb{N}$

$h \mapsto g(h) = g_{\{p,q\}}(h) =$ the minimum tiles required for a polyform in the $\{p, q\}$ Tessellation to have h holes

Polyforms as Graphs

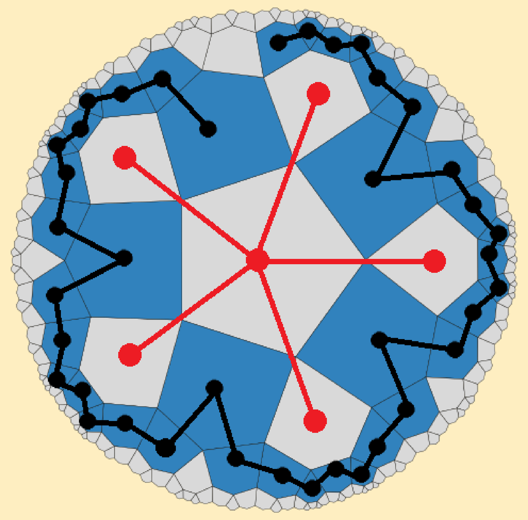


Figure: A polyform with it's dual graph overlaid in black and the hole graph overlaid in red

Dual Graph

The graph created by representing each tile of a polyform with a vertex and connecting those vertices representing adjacent tiles with edges. Useful for understanding connectivity and cyclic behaviors in polyforms.

Dual Tessellation

The dual of the $\{p, q\}$-Tessellation is the $\{q, p\}$-Tessellation. It is the same as taking the dual graph of the tessellation as a whole.

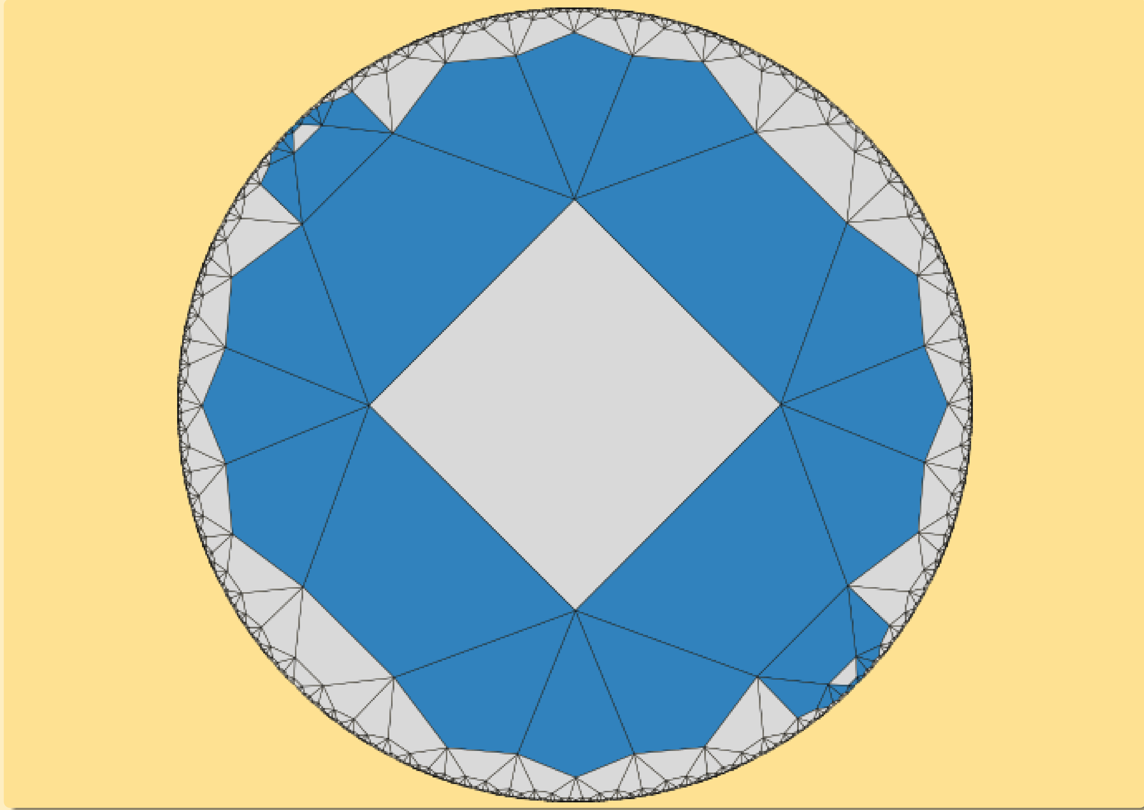
Hole Graph

The graph created by making each hole of a polyform a vertex and creating edges between those holes that touch at their corners. Useful for understanding the structure of polyforms.

Asymptotic Behavior of $g(h)$

Lemma: $\beta\left(p-1-\frac{1}{\beta}\right)h \le g(h) \le (pq-2p)h$

- The lower bound can be derived using the first lemma below, with β being the plus solution to $(p-2)x^2 - (p-2)(q-2)x + (q-2) = 0$.
- The upper bound can be found by "stacking" a simple polyform with 1 hole against itself as depicted to the right.



Conjecture: $\lim_{h\to\infty} \frac{g(h)}{h} = \beta\left(p-1-\frac{1}{\beta}\right)$

We conjecture that there exists a finite value for $\lim_{h\to\infty} \frac{g(h)}{h}$. We have shown that if it exists, it is no less than $\beta\left(p-1-\frac{1}{\beta}\right)$, where β is described above.[3].

Crystallized Polyforms

Crystallized - A polyform is crystallized when it has h holes and $g(h)$ tiles.

Let A be a polyform.

1. Lemma 1: If A has n tiles and h holes, then $h \le \frac{n(p-2)+2-\mathcal{P}^{p,q}_{\min}(n+h)}{p}$, where $\mathcal{P}^{p,q}_{\min}(n)$ is the minimum perimeter of a polyform with n tiles.
2. Lemma 2: If A has only holes with area 1, minimum outer perimeter, and the dual graph of A is acyclic then A is crystallized.
3. Conjecture: If A is crystallized (and $q > 3$) then A's dual graph is acyclic and A has only holes of area 1.

These lemmas are generalized versions of results from [2].

h	$\{3,7\}$	$\{4,5\}$	$\{4,6\}$	$\{5,4\}$	$\{5,5\}$	$\{6,4\}$	$\{7,3\}$
1	11	10	13	9	13	11	8
2	18	17	23	15	23	19	14
3	25	24	33	21	33	27	20
4	32	31	43	27			
5	39	38		33			

Table: Tiles needed for a crystallized polyform with h holes

The values in the table were computed using a modified version of depth first search on the dual of the tesselation with various maximum depths. Note the linear growth in the number of tiles required.

Example Polyforms - Crystals in $\{4, 5\}$ and $\{3, 7\}$

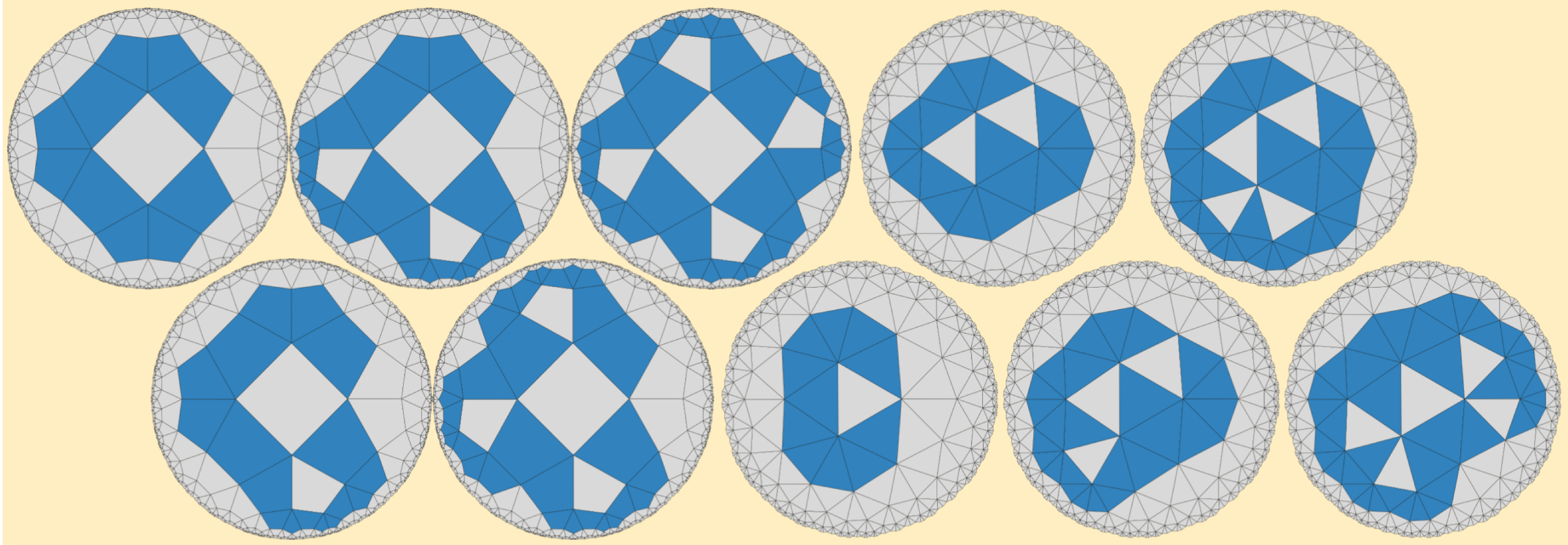
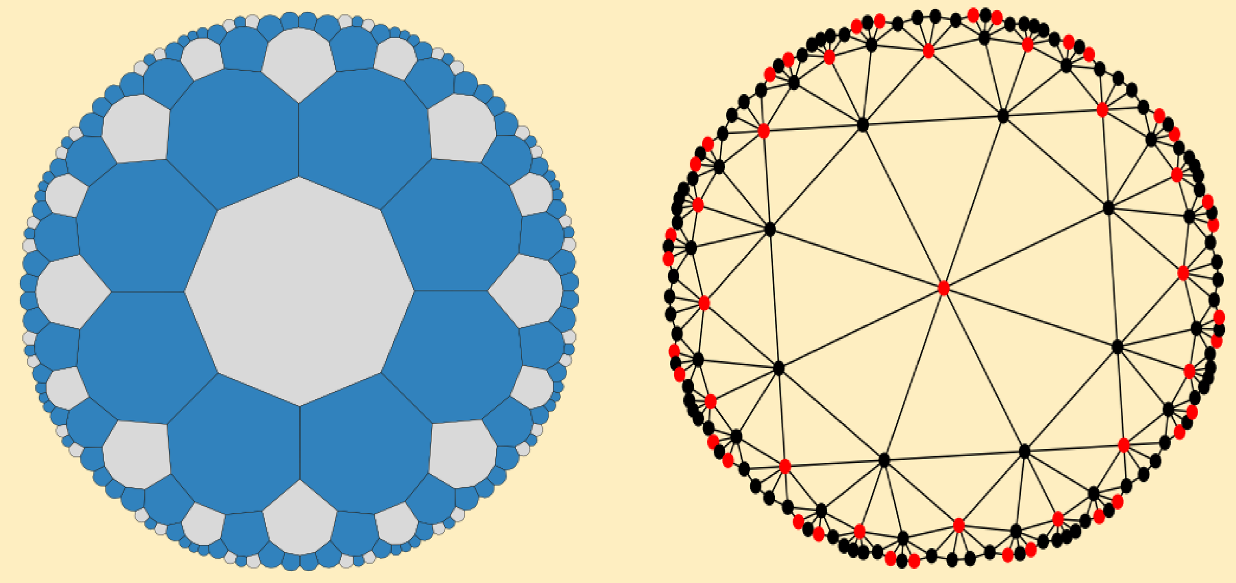


Figure: Crystallized polyforms with up to 5 holes in $\{4,5\}$ and $\{3,7\}$ respectively

The case $\{p, q\} = \{2k, 3\}$



$$g_{\{2k,3\}}(h) = \mathcal{V}^{\{2k,k\}}_{min}(h)$$

where $\mathcal{V}^{2k,k}_{min}(h)$ is the minimum number of vertices a $\{2k, k\}$ polyform with h tiles can have.

Conclusions/Future Work

In summary, we have proven many properties of crystallized hyperbolic polyforms. Furthermore, we have developed our understanding and intuition of the hyperbolic plane.
Though we are not continuing the project with MEGL, we have been working on a paper laying out our proofs and progress in detail which we expect to finish in the coming months. In the future, this work could be expanded by proving the existence and value of $\lim_{h\to\inf} g(h)$, finding closed formulae for g(h) in more cases, and improving visual representations of these polyforms to accommodate larger ones.

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References

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