Visualizing Holey Hyperbolic Polyforms

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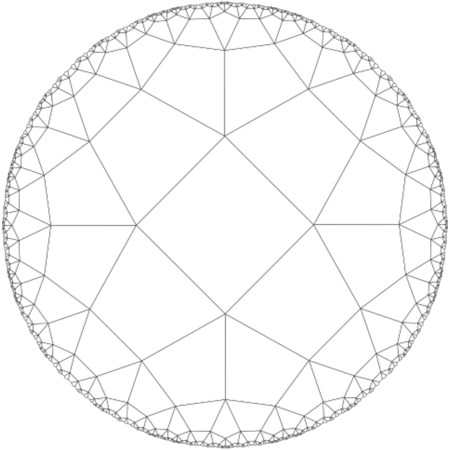
May 2, 2025

Outline

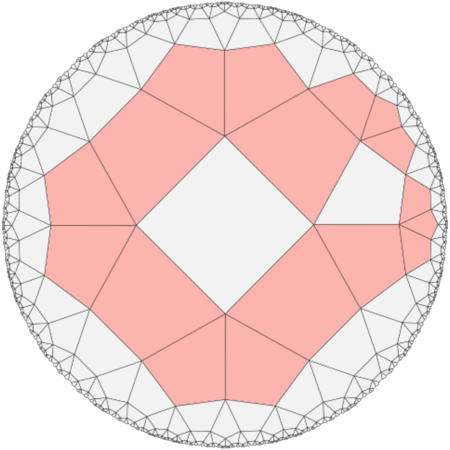
Project Background: Terms

- Hyperbolic plane: a space which differs from the Euclidean setting in several ways, the most relevant being that interior angles of regular polygons are smaller than normal and that the parallel postulate is different.
- $\{p, q\}$ Tessellation: Tiling of the plane such that at any given vertex, q regular $p-$gons meet.
  - p: the number of sides a tile in the tessellation has.
  - q: the number of tiles meeting at each vertex.
  - In all hyperbolic tessellations p and q satisfy the inequality $(p-2)(q-2) > 4$
- Polyform: A plane figure formed from regular polygons in a tessellation connected edge to edge.
- Hole: A bounded component of the complement of the polyform.

Example Polyforms: {4,5} Tessellation



Full Tessellation



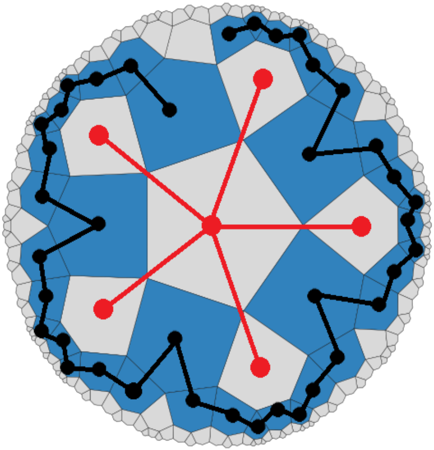
Polyform with 2 Holes

Graphs and Duality

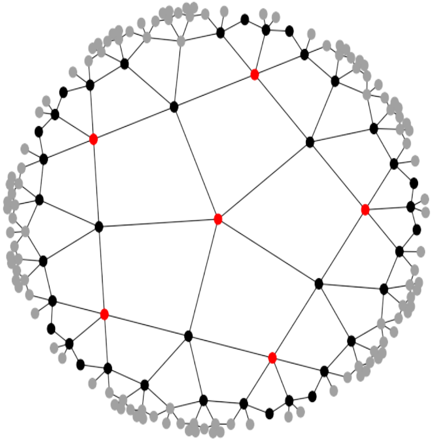
We often use graphs to represent polyforms in order to better visualize and work with them. We do this in three ways, all of which can be referred to by the term "dual".

- Dual Graph: the dual graph of a polyform is constructed by representing each tile of the polyform as a vertex and creating edges between adjacent tiles. This is the graph theoretical notion of dual.
- Dual Tessellation: The dual of the $\{p, q\}$-Tessellation is the $\{q, p\}$-Tessellation. It can be thought of as the result of the dual graph of the tessellation as a whole.
- Hole Graph: The graph created by making each hole of a polyform a vertex and creating edges between those holes that touch at their corners.

Dual Examples



A polyform in $\{5,4\}$ with its dual graph (black) and hole graph (red).



The same polyform represented in the dual tessellation with holes in red and tiles in black.

Project Objectives

Main goal: Find the function $g_{p,q}(h) = g(h)$ for the minimum number of tiles needed to have a polyform with h holes. Polyforms with h holes and $g(h)$ tiles are referred to as *crystallized*.

Sub Goals:

1. Prove various properties of crystallized polyforms, such as the optimal area of each hole
2. Find binding constants c and C (depending on p and q) such that $ch \leq g(h) \leq Ch$
3. Create examples of crystallized hyperbolic polyforms for large h.

The Role of Computation

It's basically impossible to visually work with hyperbolic polyforms without a computer.

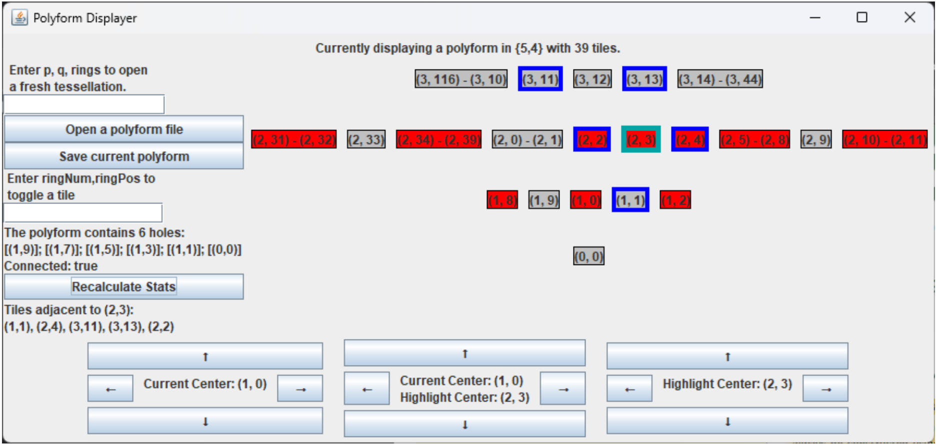
- Images are the only effective way to communicate what your polyform looks like
- After a few layers, tiles are imperceptible in the Poincaré disk or half-plane models
- To get more than a few holes, you need dozens of tiles

Computer tools can help build your intuition.

How We Used Computers

- Making examples
- Checking the distinct possible layouts of polyforms to find crystallized ones and expose patterns
- Generating tessellation plots to allow easy checks of if what we're thinking makes sense
- After a certain size, the only way to to show a polyform is to pan your perspective through the plane

Polyform Displayer Program



See https://github.com/CRoger20/megl-polyforms/

Efficient Structure

Efficiently Structured: A polyform A with n tiles and h holes is efficiently structured if it satisfies the following conditions:

1. A is acyclic: The dual graph of A is a tree - no cycles.
2. Every hole in A has area of a single tile. Equivalently, each hole is bounded by exactly p edges.
3. A has minimal outer perimeter. That is, the number of edges it has that border the unbounded region is as small as possible, or in other words, A has the smallest perimeter for a polyform with $n + h$ tiles.

Remark: A closed formula for the minimum perimeter of a $\{p, q\}$ polyform with m tiles was established by Roldán and Toalá-Enríquez in [3]

$$\mathcal{P}_{\min}^{p,q}(m) = \left(p - 2 - \frac{2}{\beta}\right)(m) + \epsilon$$

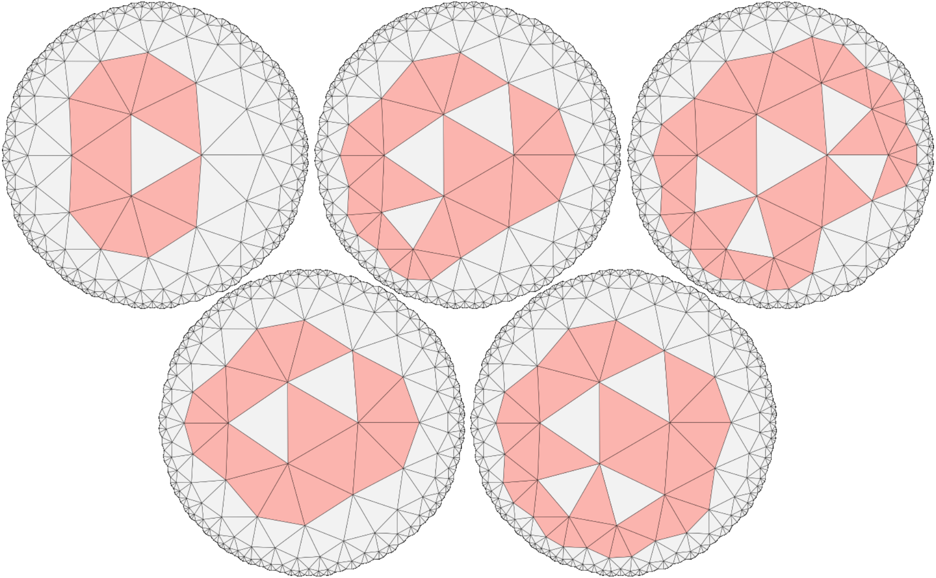
Crystallized Polyforms

Remember: We say a polyform is crystallized if it has h holes and $g(h)$ tiles.

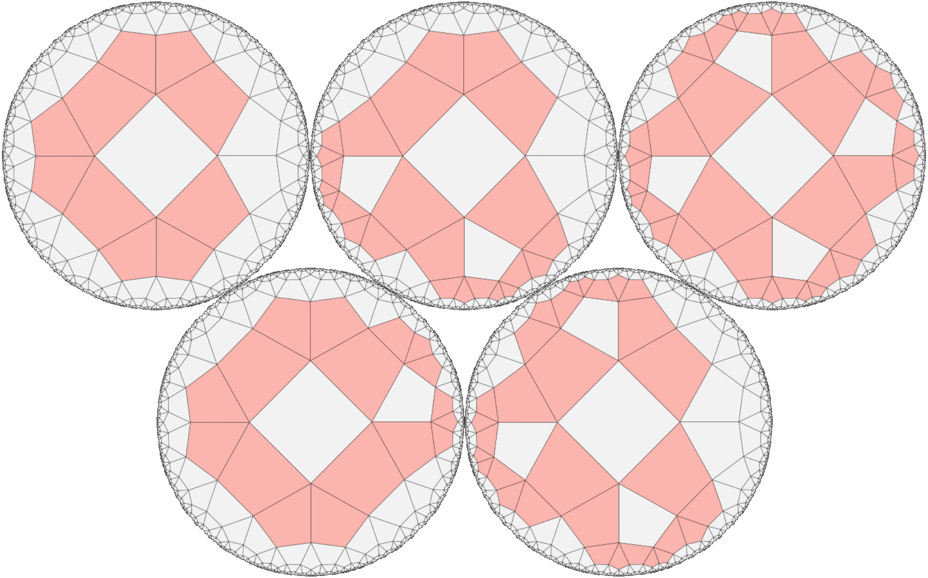
We have the following lemmas generalized from [1]. Let A be a polyform in the $\{p, q\}$ tessellation.

1. If A has n tiles and h holes, then $h \leq M(n, h) := \frac{n(p-2)+2-\mathcal{P}_{\min}^{p,q}(n+h)}{p}$.
2. If A is efficiently structured, then A is crystallized.
3. Conjecture: If A is crystallized then A has only holes with area 1. Furthermore, if it additionally holds that $q > 3$ then A is acyclic.

Crystallized Polyform Examples



$\{3, 7\}$-**Tessellation**



$\{4,5\}$-**Tessellation**

Crystallized Polyform Values

The following values in the table were computed using a modified version of depth first search on the dual of the tesselation with various maximum depths.

h	{3,7}	{4,5}	{4,6}	{5,4}	{5,5}	{6,4}	{7,3}
1	11	10	13	9	13	11	8
2	18	17	23	15	23	19	14
3	25	24	33	21	33	27	20
4	32	31	43	27			
5	39	38		33			

Table: Tiles needed for a crystallized polyform

Bounds on $g(h)$

Theorem

$\beta\left(p-1-\frac{1}{\beta}\right)h \leq g(h) \leq (pq-2p)h$

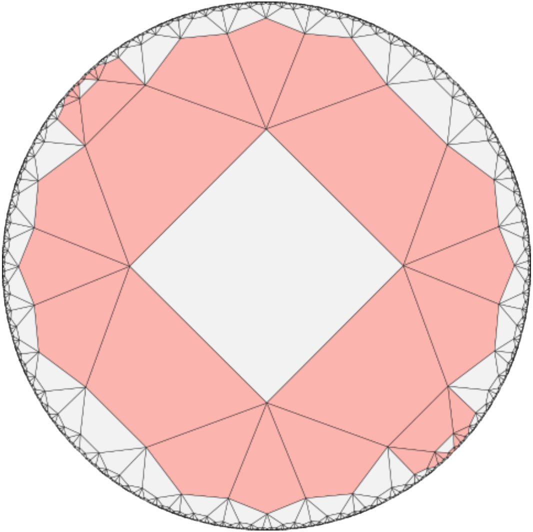
where β is the plus solution to $(p-2)x^2-(p-2)(q-2)x+(q-2)=0$.

Proof.

1. The lower bound is derived from algebra on $M(n,h)$ with $n=g(h)$
2. The upper bound can be demonstrated by creating a polyform with $(pq-2p)$ tiles and 1 hole, then "stacking" copies of it to create polyforms with $(pq-2p)h$ tiles and h holes.

□

Bounds on $g(h)$



An example of stacking the copies.

Asymptotic Behavior

We are also interested in the ratio between tiles and holes for large polyforms. We have shown that if the limit exists,

$$\lim_{h\to\infty} \frac{g(h)}{h} \geq \beta \left(p - 1 - \frac{1}{\beta} \right)$$

This can be derived from the inequality $h \leq M(n, h)$. We conjecture that not only does this limit exist, but that in crytallized polyforms $h = M(n, h)$ and thus

$$\lim_{h\to\infty} \frac{g(h)}{h} = \beta \left(p - 1 - \frac{1}{\beta} \right)$$

A Formula For $g(h)$

Theorem

$g_{2k,3}(h) = \mathcal{V}_{min}^{2k,k}(h)$
where $\mathcal{V}_{min}^{2k,k}(h)$ *is the minimum vertices in a* $\{2k, k\}$ *polyform with* h *tiles*

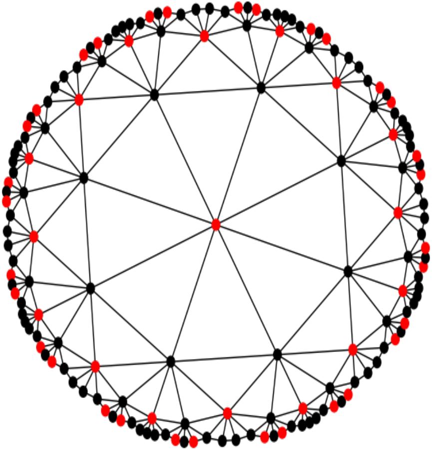
Proof.

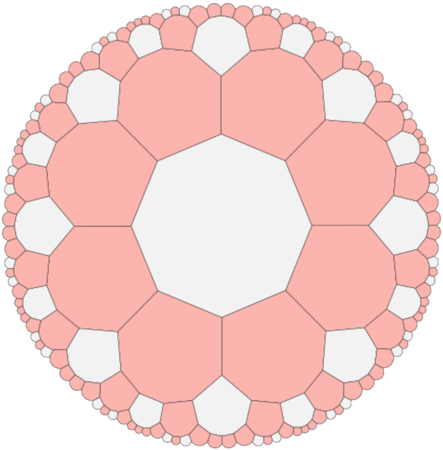
To get a $\{2k, 3\}$ polyform with $\mathcal{V}_{min}^{2k,k}(h)$ tiles and h holes:

1. Create a $\{2k, k\}$ polyform with h tiles and minimal vertices
2. For each tile in the polyform, add a vertex to its center with edges to every other vertex on the tile, forming the $\{3, 2k\}$ tessellation
3. Color the graph such that every vertex added in step 2 is red and every vertex that was already present is black.
4. Take the dual, resulting in the $\{2k, 3\}$ tessellation

The red vertices correspond to holes of the polyform, and the black vertices to its tiles. The construction has minimal black vertices and maximal red ones, so the polyform is crystallized. □

A Formula For $g(h)$





The dual graph produced in the process (left) and resultant polyform's structure (right).

Conclusions

In summary, we have proven many properties of crystallized hyperbolic polyforms. Furthermore, we have developed our understanding and intuition of the hyperbolic plane.

Though we are not continuing the project with MEGL, we have been working on a paper laying out our proofs and progress in detail which we expect to finish in the coming months. In the future, this work could be expanded by proving the existence and value of $\lim_{h\to\infty} \frac{g(h)}{h}$, finding closed formulae for $g(h)$ in more cases, and improving visual representations of these polyforms to accommodate larger ones.

Acknowledgments

We would like to thank our faculty mentors Dr. Ros Toalá and Dr. Érika Roldán as well as our graduate mentor Summer Eldridge for their guidance, direction, and resources during this project. We would also like to thank Dr. Peter Kagey for his guidance regarding enumeration of polyforms and Malin Christersson for their website that generated interactive tessellations (https://www.malinc.se/noneuclidean/en/poincaretiling.php).

References

[1] Greg Malen and Érika Roldán. "Extremal Topological and Geometric Problems for Polyominoes". In: *The Electronic Journal of Combinatorics* 27.2 (2020).

[2] Greg Malen, Érika Roldán, and Rosemberg Toalá-Enríquez. "Extremal $\{p, q\}$-animals". In: *Annals of Combinatorics* 27.1 (2023), pp. 169–209.

[3] Érika Roldán and Rosemberg Toalá-Enríquez. "Isoperimetric Formulas for Hyperbolic Animals". In: *Graphs and Combinatorics* 41.38 (2025).

[4] Manuel Schrauth et al. *The hypertiling project*. 2023. URL: https://doi.org/10.5281/zenodo.7559393.