Critical Habitat Thresholds: Phase Plane Analysis of Allee Dynamics

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Motivation

- This project examines the stability of a population in a habitat undergoing changes in size due to **perturbations**, such as climate change and human intervention.
- From our population dynamics model, we found that a population needs a minimum habitat size to remain stable. Outside this patch, the population dies off, but inside, it remains stable and survives, which is known as a **standing wave problem.**
- We are trying to find a **critical length**, such that when the habitat is larger than this size, the habitat is stable. We have proven in this project that the critical length exists given certain initial population parameters.

Model

In order to model these population dynamics, we use the following nonlinear RDE (Reaction-Diffusion Equation) in one spatial dimension:

$$u_t = Du_{xx} + f(u, H(x)), \qquad (*)$$

with an Allee growth term

$$f(u, H(x)) = -\beta u + \lambda H(x) u^2 - \gamma u^3$$

where

$$H(x) = \begin{cases} h^* \in \mathbb{R}, & \text{if } |x| \le \frac{L}{2} \\ 0, & \textit{otherwise} \end{cases}$$

with constant diffusion coefficient $D > 0$ and boundary conditions

$$\lim_{x \to \pm\infty} u(x, t) = 0.$$

Model

This model accounts for the limited resources in a large population density and for under-crowding at lower population density. We investigate steady-state solutions (where $u_t = 0$) with parameters:

$$\beta \geq 0, \quad \gamma \geq 0, \quad \lambda = 1,$$
$$D = 1, \quad h^* = 1.$$

Hence, we obtain the following ODEs for being "inside" and "outside" the habitat:

$$\begin{cases} u_{xx} - \beta u + u^2 - \gamma u^3 = 0, & \text{if } |x| \leq \frac{L}{2} \\ u_{xx} - \beta u - \gamma u^3 = 0, & \textit{otherwise} \end{cases}$$

Model

With the use of reduction of order, we obtain two first-order systems:

Inside habitat:

$u_x = v$
$v_x = \beta u - \lambda h^* u^2 + \gamma u^3$

Outside habitat:

$u_x = v$
$v_x = \beta u + \gamma u^3$

The inside and outside habitat ODEs are Hamiltonian, so we can use the energy landscape of the Hamiltonian systems to determine a critical length L^* of the habitat, under which the population collapses. This energy landscape give us a path to proving our main theorem on the existence of L.

Dynamics

Using integration, we obtain the following Hamiltonian functions from the inside and outside habitat systems, respectively:

$$
\begin{aligned}
E_g(u,v) &= \frac{v^2}{2} - \frac{\beta u^2}{2} + \frac{u^3}{3} - \frac{\gamma u^4}{4} - k, \qquad \sqrt[3]{3k} < u < \frac{1+\sqrt{1-4\gamma\beta}}{2\gamma} \\
E_b(u,v) &= \frac{v^2}{2} - \frac{\beta u^2}{2} - \frac{\gamma u^4}{4}, \qquad 0 < u < \sqrt[3]{3k}
\end{aligned}
$$

where we find that the bounds for the inside Hamiltonian function must satisfy

$$
0 < k < -\frac{\beta u^2}{2} + \frac{u^3}{3} - \frac{\gamma u^4}{4}, \quad u = \frac{1+\sqrt{1-4\gamma\beta}}{2\gamma},
$$

with parameters satisfying $0 < \gamma < \frac{2}{9\beta}$, which determine the admissible trajectories in the phase space we are studying. That is, the admissible trajectories are over the domain

$$
\left\{ u \in \mathbb{R}^+ \mid \frac{2-2\sqrt{1-4.5\gamma\beta}}{3\gamma} < u < \frac{1+\sqrt{1-4\gamma\beta}}{2\gamma}, 0 < \gamma < \frac{2}{9\beta} \right\}
$$

Dynamics

With the general formulation and its parameter restrictions, we can obtain an L-value, which is the length of the inside habitat in our ideal path of the habitat.

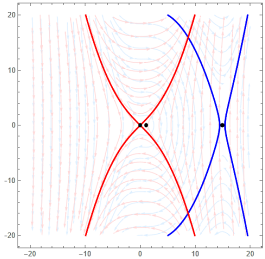


Figure: (a)

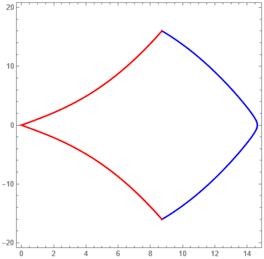


Figure: (b)

Graphs of both the inside and outside habitat with (a) all the level curves superimposed together. The ideal path (b) of the habitat is created based on where the two habitats meet. Both graphs used parameters of $\beta = 1$ and $\gamma = \frac{1}{16}$.

Existence of the critical length L

Before obtaining a value for L, we must relate L to some initial condition u_0, which starts on the outside habitat level curve. To formulate $L(u_0)$ for a given initial condition u_0, we integrate the system,

$$\frac{du}{dx} = v \tag{1}$$

$$\frac{dv}{dx} = -f(u, H(x)) \tag{2}$$

with initial conditions $(u(0), v(0)) = (u_0, v_0)$, where $v_0 = \sqrt{\beta u_0^2 + \frac{\gamma u_0^4}{2}}$ corresponds to the outside habitat energy level $E_b(u_0, v_0) = 0$. The total length L is determined by the time it takes to flow along the ideal path of the habitat. By proving L exists, we guarantee the connection between our habitats.

Existence of the critical length L

From the inside habitat Hamiltonian and using separation of variables with constant k (dependent on choice of u_0), we derive the length formula:

$$L(u_0) = 2\int_{u_0}^{u_1(u_0)} \frac{1}{\sqrt{\beta\tau^2 - \frac{2}{3}\tau^3 + \frac{1}{2}\gamma\tau^4 + 2k(u_0)}}\,d\tau$$

This formulation paves the way for proving our main theorem.

Theorem

Theorem (Habitat Collapse Criterion)

Consider the the reaction-diffusion equation with $u : \Omega \times [0, \infty) \to [0, \infty)$ *and parameters* $\beta > 0$, $h^* = \lambda = 1$, *and* γ *satisfying* $0 < \gamma < \frac{2}{9\beta}$. *Let* $L : (0, u_+) \to (0, \infty)$. *Then there exists a critical length* $L^* > 0$ *such that if* $L > L^*$, *then the habitat has exactly two steady-state pulse solutions; otherwise, there are zero for* $L < L^*$.

Theorem

To prove the existence of a critical length $L^* > 0$, we must satisfy three conditions:

(1) $\lim_{u_0 \downarrow 0} L(u_0) = \infty$
(2) $\lim_{u_1(u_0) \uparrow u_+} L(u_1(u_0)) = \infty$
(3) $L''(u_0) > 0$ for all $u_0 \in (0, u_+)$

We will go through the highlights of the proof (see write-up for more details).

*Note here that $u+$ is the maximum fixed point value for the inside Hamiltonian system, i.e. $u_+ = \frac{1+\sqrt{1-4\gamma\beta}}{2\gamma}$

(1) $\lim_{u_0 \downarrow 0} L(u_0) = \infty$

From the integrand, we can find the lower bound for

$$\frac{1}{\sqrt{\beta\tau^2 - \frac{2}{3}\tau^3 + \frac{\gamma}{2}\tau^4 + \frac{2}{3}u_0^3}},$$

Since our bounds of integration are from u_0 to $u_1(u_0)$, then $u_0 \leq \tau \leq u_1$, which means

$$\frac{1}{\sqrt{\beta\tau^2 - \frac{2}{3}\tau^3 + \frac{\gamma}{2}\tau^4 + k(u_0)}} \geq \frac{1}{\tau\sqrt{\beta + \frac{\gamma}{2}u_1^2}}.$$

Next, integrating both sides from u_0 to $u_1(u_0)$ we get

$$\int_{u_0}^{u_1(u_0)} \frac{1}{\sqrt{\beta\tau^2 - \frac{2}{3}\tau^3 + \frac{\gamma}{2}\tau^4 + k(u_0)}} d\tau \geq \int_{u_0}^{u_1(u_0)} \frac{1}{\tau\sqrt{\beta + \frac{\gamma}{2}u_1(u_0)^2}} d\tau.$$

(1) $\lim_{u_0 \downarrow 0} L(u_0) = \infty$

We notice that the right of our inequality goes to ∞ as $u_0 \downarrow 0$.
With that, since

$$\int_{u_0}^{u_1(u_0)} \frac{1}{\sqrt{\beta\tau^2 - \frac{2}{3}\tau^3 + \frac{\gamma}{2}\tau^4 + k(u_0)}} d\tau \geq \int_{u_0}^{u_1(u_0)} \frac{1}{\tau\sqrt{\beta + \frac{\gamma}{2}u_1(u_0)^2}} d\tau,$$

then

$$\int_{u_0}^{u_1(u_0)} \frac{1}{\sqrt{\beta\tau^2 - \frac{2}{3}\tau^3 + \frac{\gamma}{2}\tau^4 + k(u_0)}} d\tau \to \infty$$

as $u_0 \to 0$. This proves condition (1).

(2) $\lim_{u_1(u_0)\uparrow u_+} L(u_1(u_0)) = \infty$

To prove this condition, we will use the change of variables $\tau = u_1 - s^2$ and

$$f(s) = \beta(u_1 - s^2)^2 - \frac{2}{3}(u_1 - s^2)^3 + \frac{\gamma}{2}(u_+ - s^2)^4 + 2k(u_0).$$

When reduced using our previous equations and letting s approach u_+, we get

$$f(s) \sim (-2\beta u_1 + 2u_1^2 - 2\gamma u_1^3)s^2.$$

We can then integrate this, letting $s_0 = \sqrt{u_1 - u_0}$, $s_1 = \sqrt{u_1 - u_1} = 0$ and $d\tau = -2sds$. We then obtain

$$\int_{s_0}^{s_1} \frac{-2}{\sqrt{f(s)}} ds = \frac{2s_0}{\sqrt{-2\beta u_1 + 2u_1^2 - 2\gamma u_1^3}}.$$

As you take the limit of this integral as $u_1(u_0) \uparrow u_+$, the denominator approaches 0. Thus, $\lim_{u_1\uparrow u_+} L(u_1(u_0)) = \infty$, proving condition (2).

(3) $L''(u_0) > 0$ for all $u_0 \in (0, u_+)$

For condition (3), we establish that $L(u_0)$ is well-defined on $(0, u_+)$. Using a change of variables $\tau = u_1 - s^2$, we can rewrite

$$L(u_0) = 2\int_{s_0}^{0} \frac{-2}{\sqrt{f(s)}} ds,$$

where $f(s) = \beta(u_1 - s^2)^2 - \frac{2}{3}(u_1 - s^2)^3 + \frac{\gamma}{2}(u_1 - s^2)^4 + k(u_0)$. Following a similar argument to the previous conditions, we show $f(s)$ does not vanish when $u_1 \neq u_+$.

(3) $L''(u_0) > 0$ for all $u_0 \in (0, u_+)$

Let us now define the function from the denominator on the length function integrand as

$$V(u, u_0) = \beta u^2 - \frac{2}{3}u^3 + \frac{\gamma}{2}u^4 + k(u_0). \tag{3}$$

The partial derivative

$$\frac{\partial V}{\partial u_0}(u, u_0) = \frac{\partial k}{\partial u_0} = 2u_0^2 - 2\beta u_0 - 2\gamma u_0^3 \tag{4}$$

changes signs exactly once in $(0, u_+)$, at $\hat{u}_0$ splitting it into two regions. We define $\hat{u}_0$ to be the maximum of the cubic Allee growth model, which changes depending on γ, β, and u_0.

Region 1: $0 < u_0 < \hat{u}_0$

In this region, $\frac{\partial k}{\partial u_0} < 0$, Fixing u, as u_0 increases, $k(u_0)$, $V(u, u_0)$ decreases. Consider the function $g(u_0) = \frac{1}{\sqrt{V(u,u_0)}}$ for fixed u. Since $V(u, u_0)$ is decreasing, $g(u_0)$ increases. Computing the second derivative of g:

$$\frac{d^2 g}{du_0^2} = \frac{3(\frac{\partial V}{\partial u_0})^2}{4V^{5/2}} - \frac{\frac{\partial^2 V}{\partial u_0^2}}{2V^{3/2}} \tag{5}$$

The first term is always positive. For small u_0, $\frac{\partial^2 V}{\partial u_0^2} = 4u_0 - 2\beta - 6\gamma u_0^2$ is negative, making the second term positive as well. Thus, $g(u_0)$ is strictly convex. As u_0 increases, the integrand increases pointwise while the domain of integration shrinks. These combined effects ensure that $L'(u_0)$ is strictly increasing in this region.

Region 2: $\hat{u}_0 < u_0 < u_+$

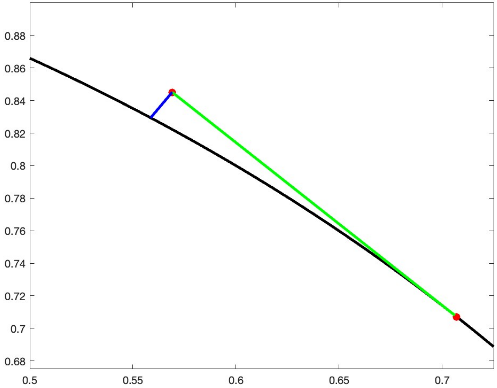
In this region, $\frac{\partial k}{\partial u_0} > 0$, so $V(u, u_0)$ increases as u_0 increases for fixed u.
Using the change of variables $u = u_1(u_0) - s^2$, the integral transforms to:

$$L(u_0) \approx \frac{4}{\sqrt{\left|\frac{\partial V}{\partial u}(u_1(u_0), u_0)\right|}} \cdot \sqrt{u_1(u_0) - u_0} \tag{6}$$

As u_0 approaches u_+, $\frac{\partial V}{\partial u}(u_1(u_0), u_0)$ approaches zero at a quadratic rate, while $\sqrt{u_1(u_0) - u_0}$ approaches zero at only a square root rate. This asymptotic behavior ensures that $L''(u_0) > 0$ throughout Region 2.
Since $L(u_0)$ is strictly convex in both regions, and the second derivative is continuous across the boundary point $\hat{u}_0$, we conclude that $L(u_0)$ is strictly convex on the entire interval $(0, u_+)$.

Numerical Computation

To systematically explore the relationship between u_0 and $L(u_0)$, we implement a pseudo-arclength continuation method that allows us to trace the full $L(u_0)$ curve efficiently, even near turning points where standard continuation (such as Newton's method) might fail.



Numerical Computation

Mathematically, if (u_k, L_k) is the current solution point with tangent vector $\mathrm{t}_k = (t_u, t_L)$, the predictor step gives:

$$u_{k+1}^{(0)} = u_k + \Delta s \cdot t_u \quad (7)$$

$$L_{k+1}^{(0)} = L_k + \Delta s \cdot t_L \quad (8)$$

where Δs is the arclength step size. The corrector iterations then solve:

$$F(u, L) = 0 \quad (9)$$

$$(u - u_{k+1}^{(0)})t_u + (L - L_{k+1}^{(0)})t_L - \Delta s = 0 \quad (10)$$

where $F(u, L) = L - \mathcal{L}(u)$ represents the constraint that L must equal the computed integration time $\mathcal{L}(u)$ for the trajectory starting at u.

Numerical Computation

We used MATLAB's ODE solver, `odes15s`, using very small tolerances ($\text{RelTol} = 10^{-12}$, $\text{AbsTol} = 10^{-14}$). This ensures that obtain we solutions that are very accurate.

The Jacobian for Newton's method is approximated using finite differences:

$$\frac{\partial \mathcal{L}}{\partial u} \approx \frac{\mathcal{L}(u + \delta u) - \mathcal{L}(u)}{\delta u} \tag{11}$$

Numerical Computation

The critical length L^* corresponds to the global minimum of the $L(u_0)$ curve. The steps to calculate the minimum length is as follows.

1. Computing the $L(u_0)$ curve across a wide range of u_0 values
2. Locating the turning points where $\frac{dL}{du_0} = 0$
3. Identifying the global minimum among these turning points

Numerical Computation

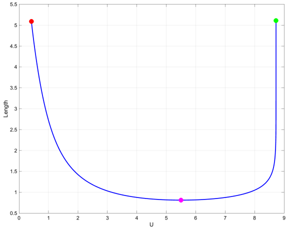


Figure: Bifurcation diagram of habitat length L versus initial condition u_0

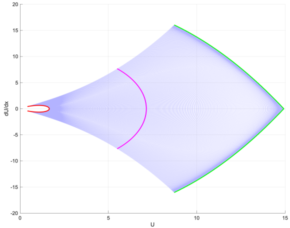


Figure: Phase plane trajectories

Numerical Computation

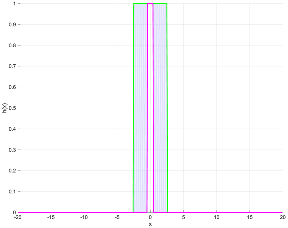


Figure: Habitat function profiles

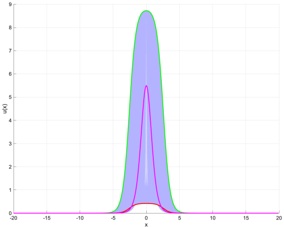


Figure: Spatial profiles of the population density using a smoothed out Habitat function

Conclusion/Future Work

- Show that our length function implies two standing waves in the system –one stable and one unstable –and further investigate the unstable wave.
- Continuing past the steady-state of the RDE, exploring the traveling wave problem will be helpful in understanding the Allee effect in greater detail.
  - Studying the traveling wave problem is done by adopting a moving-frame coordinate system, redefining the habitat function as $H(x - ct)$ with speed c. To do this, we would use similar methods to find a critical speed c^*

References

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