Critical Habitat Thresholds: Phase Plane Analysis of Allee Dynamics

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Introduction

Motivated by problems in ecology, we investigate:

1) The survival of populations within a habitat patch and establish criteria for determining minimum habitat lengths.
2) The "standing wave problem," where habitat collapse occurs when the patch becomes too small, using reaction-diffusion equations and dynamical systems theory.
3) Pulse-like standing waves in an Allee-type growth model through numerical methods, aiming to determine precise length thresholds for population persistence.

The Model

In order to model these population dynamics, we use the following nonlinear RDE (Reaction-Diffusion Equation) in one spatial dimension:

$$u_t = Du_{xx} + f(u, H(x)), \qquad (*)$$

with an Allee growth term

$$f(u, H(x)) = -\beta u + \lambda H(x) u^2 - \gamma u^3$$

where

$$H(x) = \begin{cases} h^* \in \mathbb{R}, & \text{if } |x| \leq \frac{L}{2} \\ 0, & \textit{otherwise} \end{cases}$$

with constant diffusion coefficient $D > 0$ and boundary conditions

$$\lim_{x \to \pm\infty} u(x, t) = 0.$$

This model accounts for the limited resources in a large population density and for under-crowding at lower population density. We investigate steady-state solutions (where $u_t = 0$) with parameters:

$$\beta \geq 0, \quad \gamma \geq 0, \quad \lambda = 1,$$
$$D = 1, \quad h^* = 1.$$

Hence, we obtain the following ODEs for being "inside" and "outside" the habitat:

$$\begin{cases} u_{xx} - \beta u + u^2 - \gamma u^3 = 0, & \text{if } |x| \leq \frac{L}{2} \\ u_{xx} - \beta u - \gamma u^3 = 0, & \textit{otherwise} \end{cases}$$

With the use of reduction of order, we obtain two first-order systems:

Inside habitat:	**Outside habitat:**
$u_x = v$	$u_x = v$
$v_x = \beta u - \lambda h^* u^2 + \gamma u^3$	$v_x = \beta u + \gamma u^3$

From the inside habitat ODE, we establish a critical length L^*, where $\frac{L^*}{2}$ represents the minimum half-length required for population survival. When $L < L^*$, the population cannot persist.

Dynamics

Separation of Energy Levels

For all $u > 0$, the energy levels of the inside and outside habitat are given as graphs over the u variable which satisfy:

$$|v_{\text{in}}(u)| < |v_{\text{out}}(u)|$$

where

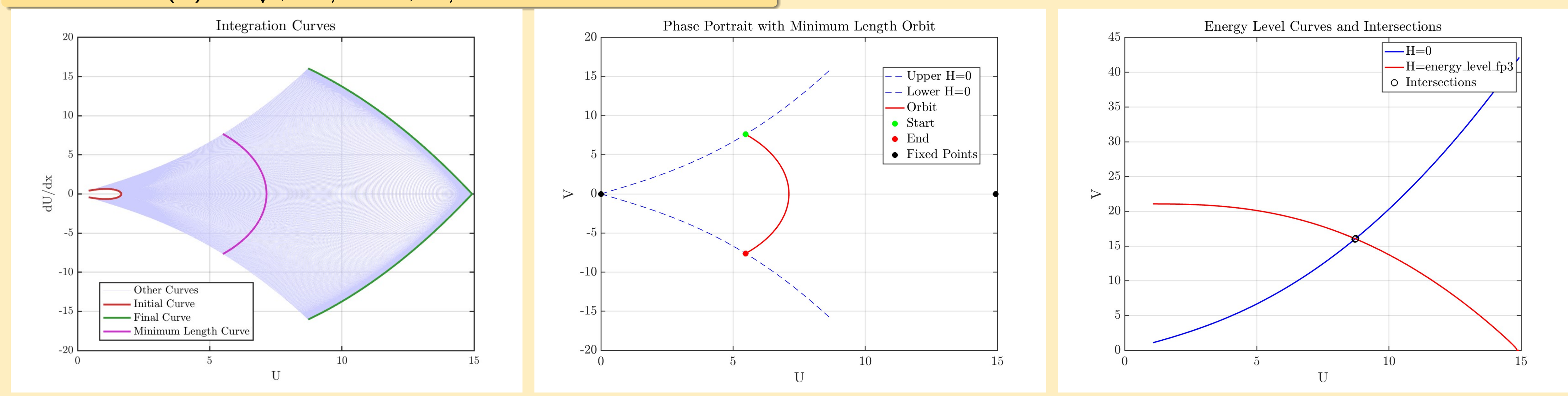
$$v_{\text{in}}(u) = \sqrt{\beta u^2/2 - u^3/3 + \gamma u^4/4} \quad \text{and}$$
$$v_{\text{out}}(u) = \sqrt{\beta u^2/2 + \gamma u^4/4}$$

Energy at Fixed Points

When $0 < \gamma < \frac{2}{9\beta}$, the energy values at the fixed points satisfy:

$$E(0,0) = 0, \quad E(u_-, 0) < 0, \quad E(u_+, 0) > 0$$

This energy landscape is necessary for the existence of pulse-type solutions.



(a) Phase space trajectories (b) Phase portrait of minimum length orbit (c) Intersection of inside and outside habitat

Critical Length of Habitat

The habitat collapses when its length falls below a critical value L^*. This length L^* corresponds to the global minimum of the $L(u_0)$ curve.

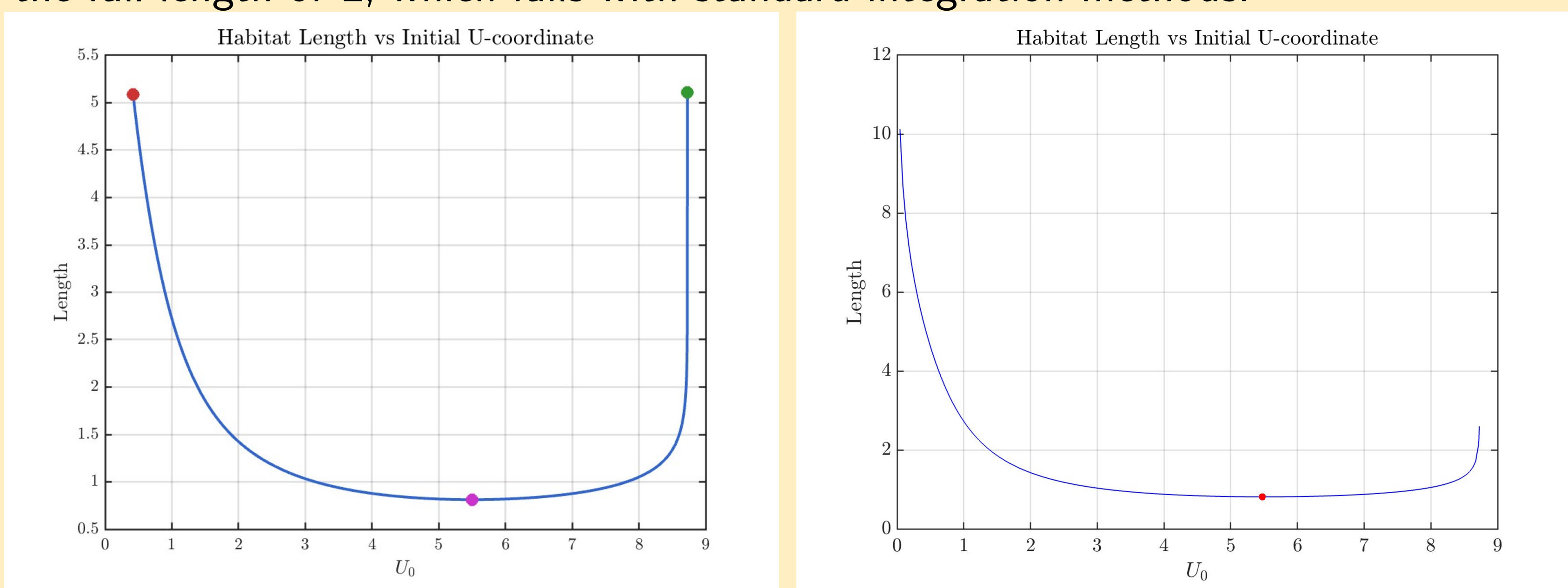
Theorem

Consider the the reaction-diffusion equation with $u : \Omega \times [0, \infty) \to [0, \infty)$ and parameters $\beta > 0$, $h^* = \lambda = 1$, and γ satisfying $0 < \gamma < \frac{2}{9\beta}$. Let $L : (0, u_+) \to (0, \infty)$ be at least twice differentiable. Then there exists a critical length $L^* > 0$ such that if $L > L^*$, then the habitat has exactly two steady-state pulse solutions; otherwise, there are zero for $L < L^*$.

Corollary

For any $L > L^*$, these two steady-state solutions occur at the intersections of the horizontal line at height L with the $L(u_0)$ curve: a stable pulse enabling population persistence and an unstable pulse defining the minimum viable population threshold.

We computed the critical length of the habitat numerically by using pseudo-arclength continuation. Using this method allowed us to trace the full length of L, which fails with standard integration methods.

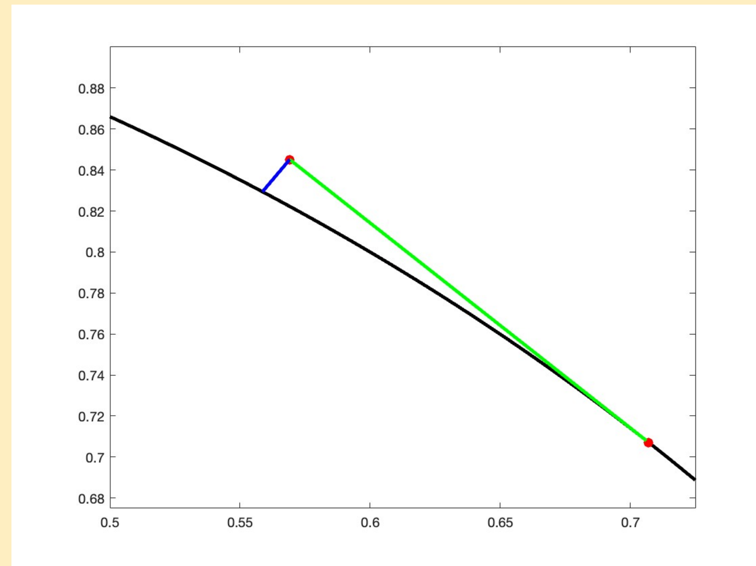


(d) L computed with continuation (e) L computed without continuation

Pseudo-Arclength Continuation Psuedocode

Input: Number of steps n, step size ds, initial point x_0
Output: Solution matrix containing continuation curve

Initialize solution matrix with x_0 as first column;
for $i = 1$ **to** n **do**
  Compute $df = \nabla F(x_0)$; // Gradient of curve
  Compute tangent vector at x_0;
  $w = x_0 + ds \cdot$ tangent ; // Predictor step
  Construct augmented Jacobian DF with df and tangent constraint;
  $x_1 = x_0 - DF \backslash F$; // Corrector step
  Append x_1 to solution matrix;
  $x_0 \leftarrow x_1$;
end
return *Solution matrix*



(f) Pseudo-arclength continuation visual

Conclusions/Future Work

In future work, we will investigate pulse stability using the Evans Function or Sturm-Liouville theory, and explore how wave speed affects critical habitat length and persistence state bifurcation structure.

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