geometry of random surfaces: percolation in higher dimensions

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Introduction

Our project studies ***percolation*** in different spaces. We're specifically interested in ***minimal blocking components*** at and around the ***critical probability threshold***, which can tell us a lot about the geometry of a space at criticality. Finding the critical threshold and minimal components require a variety of algorithms in different dimensions. Specifically, we used random graph techniques and the ***max-flow min-cut theorem*** to generate our spanning surfaces.

Definitions

Percolation describes flow through a permeable material. We simulate percolation by randomly adding edges to a d-dimensional ***lattice graph*** $G \subset \mathbb{Z}^d$ to study ***minimal blocking components*** or ***blocking surfaces*** of G. Starting with an edgeless lattice graph G,

1. Add an edge of G with probability p.
2. If there is a ***blocking surface*** C (that connects "opposite" sides of G), continue; otherwise, return to the previous step.
3. Delete the youngest edge of C.
4. In the ***dual graph*** G^* of G, find the blocking surface C^* dual to C.
5. Find the subgraph of C^* that crosses G^* using the fewest number of edges.

The ***critical probability*** $p_c(G)$ is the probability at which, by adding edges with probability p_c, a blocking surface is guaranteed to exist somewhere in the infinite grid. If $G = \mathbb{Z}^2$, then $p_c(G) = 1/2$. Our computational experiments gave us this same probability by dividing the number of edges added when a vertical crossing is created by the total number of edges in G. In higher dimensions and other spaces, the critical probability differs, and sometimes is only known experimentally. Many critical probabilities are still unknown. The figures at right show the minimal blocking surface of a percolation whose probability p is approaching p_c.

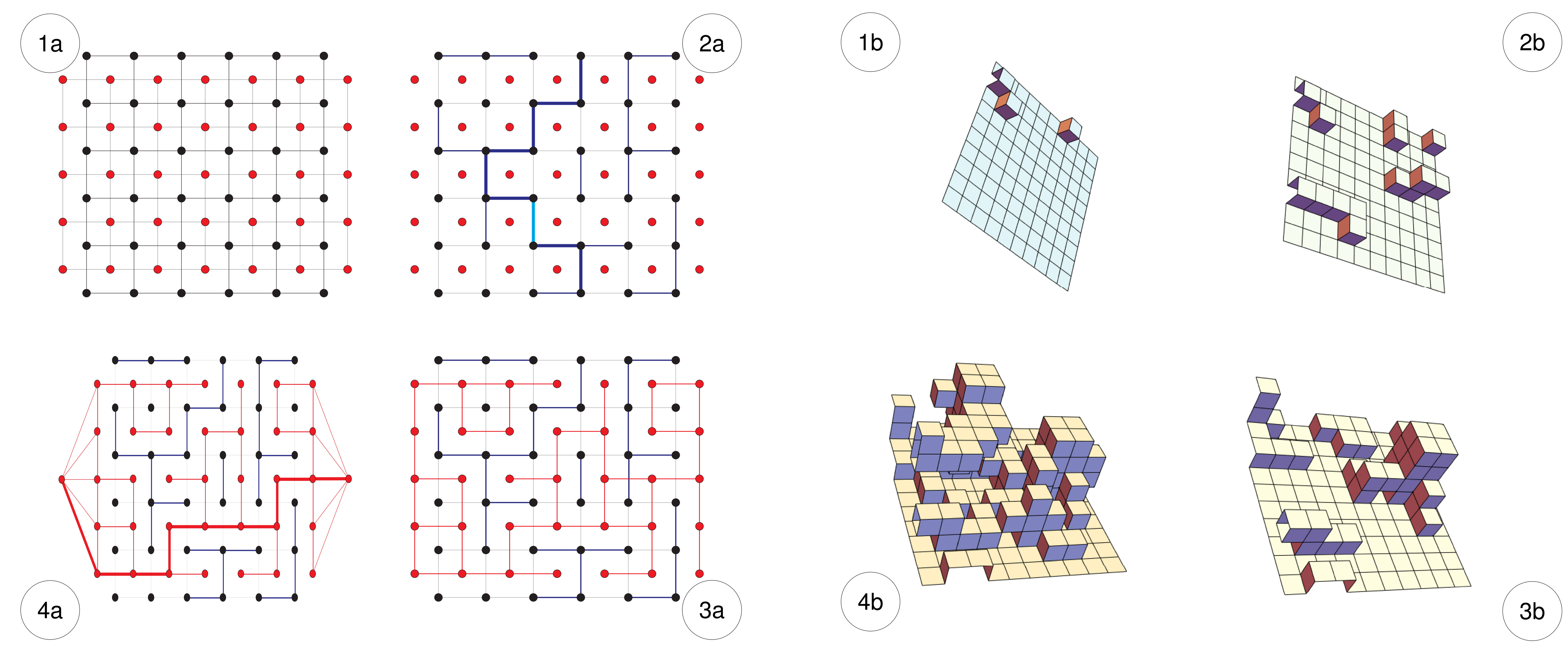
Minimal blocking components or ***blocking surfaces*** are sets of edges, squares, or higher dimensional tiles that cut the entire graph in two. We are interested in finding the path which achieves this in as little edges, squares, cubes, etc as possible. Figures 3a and 1b-4b are all examples of minimal blocking surfaces.

The ***max-flow min-cut theorem*** states that in a flow network, the maximum flow through the network from a source node to a sink node is equal to the combined capacity of the minimal set of edges you'd have to block to cut the source off from the sink. We use this theorem to compute minimal blocking surfaces. We can use a variety of well-known max-flow algorithms, and in doing so we are also solving for the minimum spanning surface through the system.

Higher Dimensions

We can apply these ideas to m-dimensional cells in an n-dimensional lattice: instead of adding *edges* randomly, we add in m-dimensional hypercubes. Our "blocking surface" is then an object of dimension $(n-m)$ separating two opposite $(n-1)$-dimensional "faces." This idea of a blocking surface is less intuitive than the 1-D case, where the surface actually blocks a 1-D path, but still represents a subset that divides the space into two subspaces. Topologically, these surfaces are representatives of the $(n-m)^{\text{th}}$ homology group.

Minimal Crossing Surfaces



Future Work

In the future we would would like to find critical values in higher-dimensional percolation models. Our naïve algorithm's efficiency decreases with system complexity, so we turn to topological techniques like ***persistent homology*** to find blocking surfaces. Some models of interest are the ***permutohedral lattice*** in four dimensions, ***two-dimensional percolation in the four-dimensional cubical lattice***, and ***two-dimensional percolation in the three-fold cubical torus***. To guide our study, we have a few questions in mind:

1. ***What is the critical threshold for two-dimensional percolation in the four-torus?*** If this lines up with what we already know, can we apply the same techniques to other, more complex systems?
2. How can we ***measure geometric properties*** of blocking surfaces?
3. ***Are these geometric values consistent across different systems?*** Do they depend on the space, the percolation model, the system size? Do ideas from classical percolation generalize to higher dimensions?

References

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