Visualizing Holey Hyperbolic Polyforms

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Introduction

It is difficult to work with things that are hard to represent. Hyperbolic polyforms are no exception. In our project, we seek to find useful ways to represent polyforms with large numbers of tiles and holes both visually and algebraically, and to find other properties of polyforms, such as a functions relating the number of tiles and holes in a polyform.

Definitions

$\{p, q\}$ Tessellation

1. Tessellation - a covering of the plane with tiles
2. p - the number of sides each tile has
3. q - the number of tiles meeting at each vertex

Our project deals only with hyperbolic tessellations, which satisfy the equation $(p-2)(q-2) > 4$

Polyform

A figure constructed out of a collection of joined tiles from the tessellation.

These are colored red in the images.

Hole

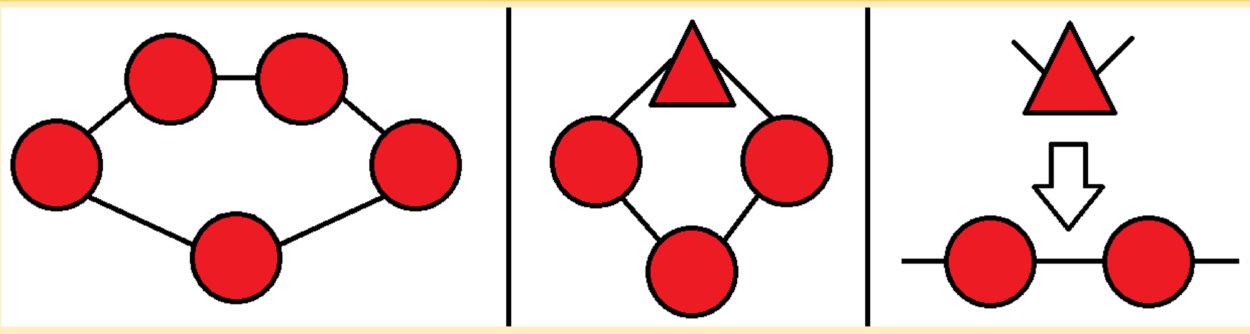
In simple terms, a hole is part of the tessellation that's completely surrounded by polyform. More formally, a hole is a bounded component of the complement of the polyform in the tessellation. These are colored gray in the images.

The Function $g(h)$

$g(h) : \mathbb{N} \to \mathbb{N}$

$h \mapsto$ the minimum tiles required for a polyform to have h holes

A Graph Representation of Polyforms

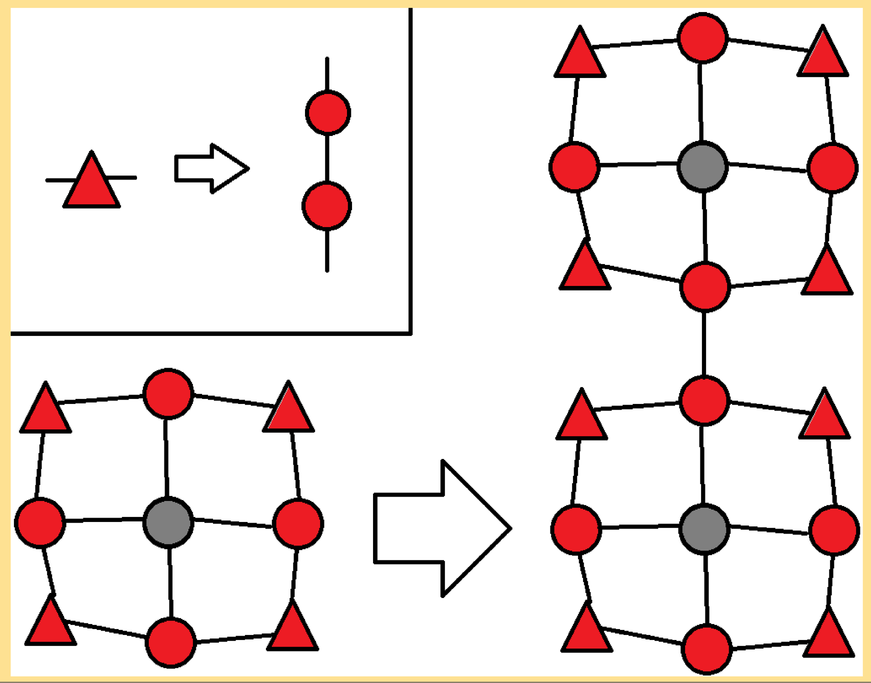


We represent polyforms as a graph by modifying the dual of a sub-graph of the tessellation. Vertices (drawn as circles) represent tiles and edges show adjacency. Groups of adjacent vertices are then replaced with a new vertex that is labeled to show which type of replacement was done, and special edges are added to show orientation. This allows more information to be displayed before the exponential growth of hyperbolic space becomes overwhelming.

In the example above, the two left polyforms are the same, with the key on the right showing the replacement that was done.

The Growth of $g(h)$ is Linear

As depicted on the right, a polyform with h holes can always be constructed using $h(pq-2p)$ tiles by repeatedly placing a structure that has one hole next to copies of itself. Thus, while not necessarily optimal, $h(pq-2p)$ serves as an upper bound. Additionally, because each hole in a subset of the $\{p,q\}$-tessellation requires a minimum of p edges to bound it, a polyform with h holes must have hp edges. Since each tile of the polyform has exactly p edges, a polyform with h holes has at least h tiles. Since $g(h)$ has both a linear upper bound and lower bound, it must have linear growth.



Computed Minimum Tiles for h Holes

h	{3,7}	{4,5}	{4,6}	{5,4}	{5,5}	{6,4}	{7,3}
1	11	10	13	9	13	11	8
2	18	17	23	15	23	19	14
3	25	24	33	21	33	27	20
4	32	31	43	27			
5	39	38		33			

Table: Tiles needed for a polyform with h holes

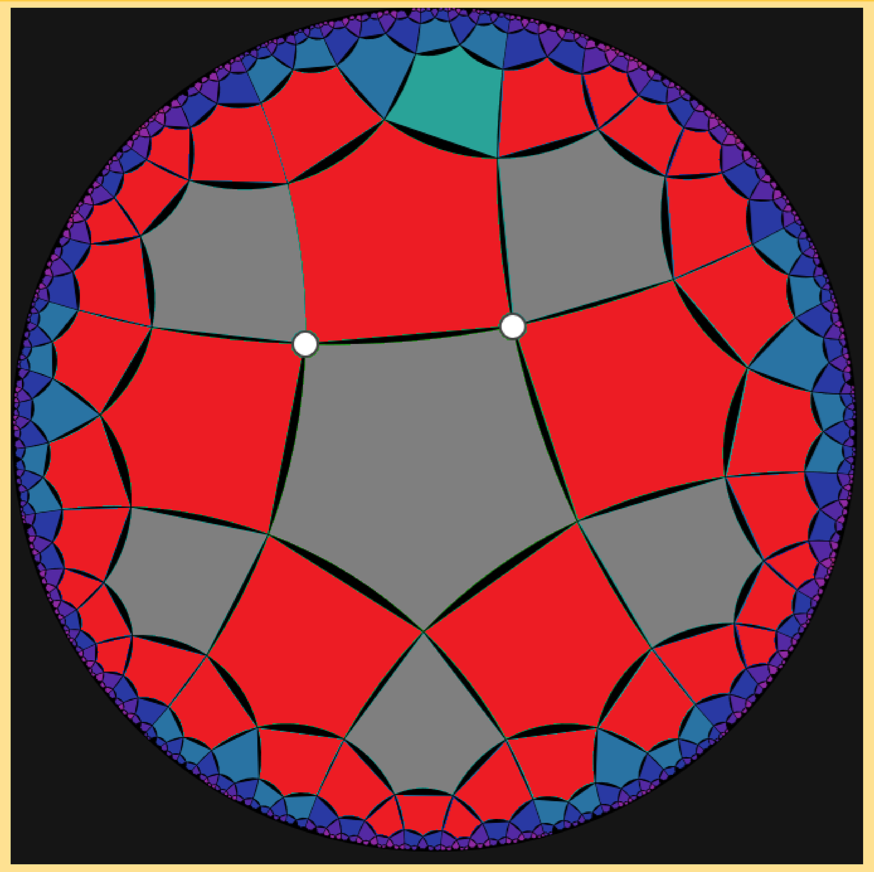
The values in the table were computed using a deterministic version of the algorithm depicted to the right with various values of N. Tiles selected for labeling were determined by DFS without backtracking.

```
Algorithm 1: Find Holeyest Polyform
Input: N, R, P, Q
Create sub-graph of {P,Q} tessellation containing
every tile within R rings of the origin
Label a tile as part of the polyform
while Fewer than N tiles are labeled do
    if Another hole can be created then
        Choose and label the best tile
    else
        Exit Loop
Output: The labeled tiles as a polyform
```

Though this algorithm is O(n) time, the deterministic version is O(p^n) time, which quickly becomes infeasible.

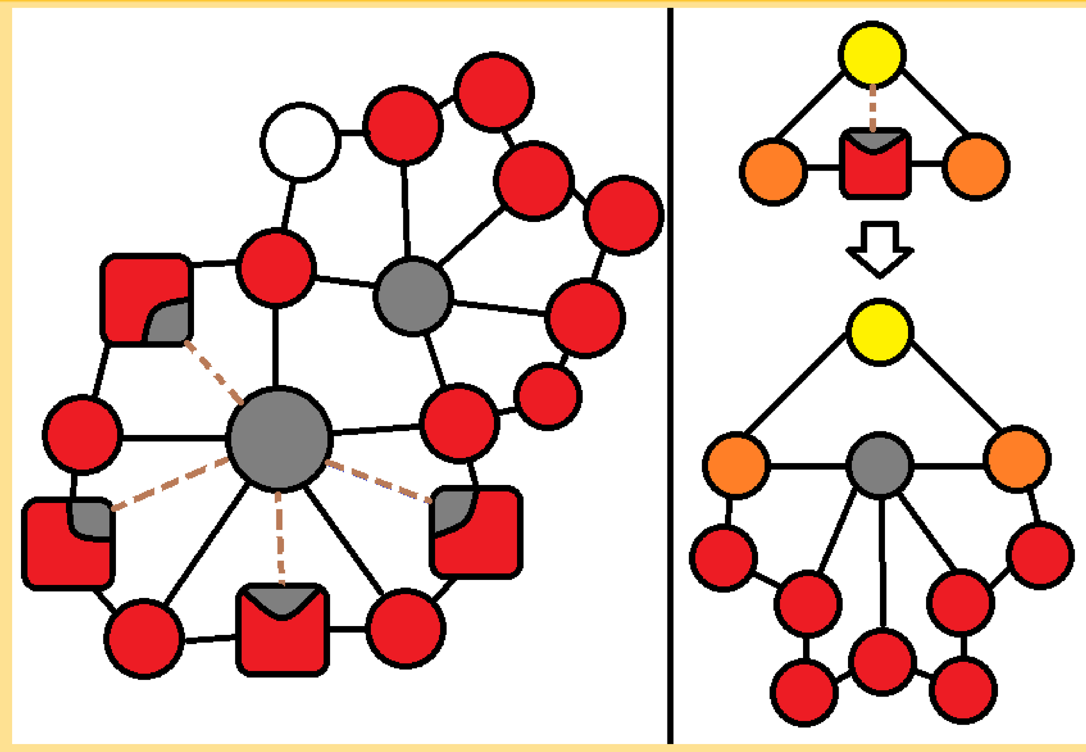
Example Polyforms

A polyform in $\{5,4\}$ with 39 tiles and 6 holes



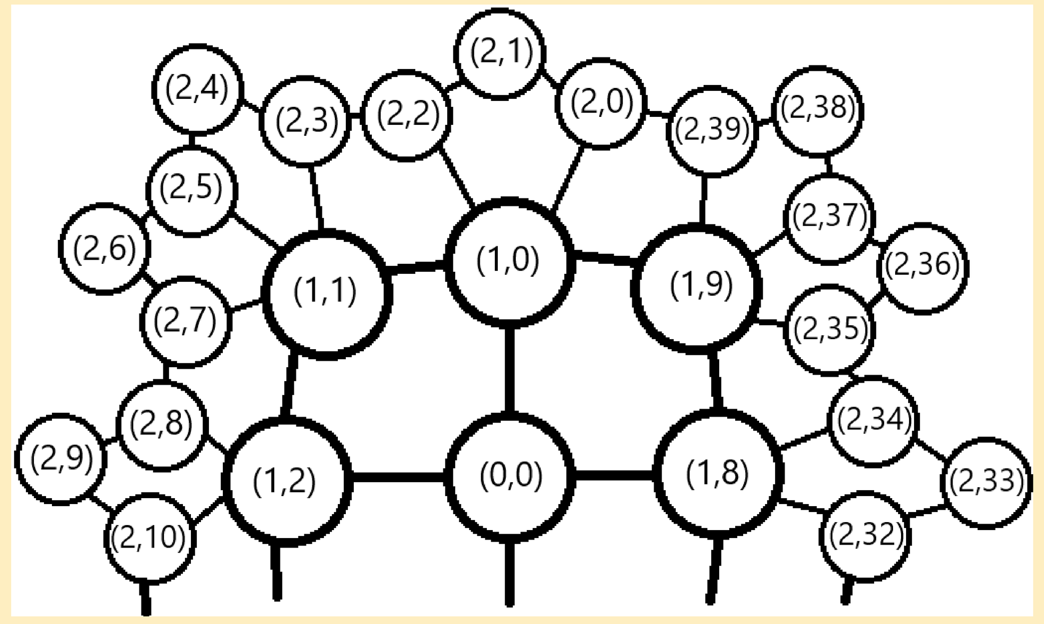
A recolored version of an image from [1].

The same polyform represented as a graph



The left side is the polyform. The right side is a key for the special square symbol.

Tile Coordinates



We use a coordinate system in which we choose a center tile as (0,0) and label other tiles based on "rings" about the center, with (a,b) being the bth tile in the ath ring.

Conclusions/Future Work

In summary, we have proven a linear growth rate for $g(h)$, computed the values of $g(h)$ for small h in various tessellations, and developed representations of polyforms as graphs with symbols to abstract groups of vertices.

Next semester, we will continue to explore representations of polyforms (especially as graphs and as elements of groups) and establish more properties of $g(h)$ such as more precise bounds, and if possible, a closed formula.

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References

[1] Malin Christersson. *Non-Euclidean Geometry: Interactive Hyperbolic Tiling in the Poincaré Disc.* Accessed: November 30, 2024. 2018. URL: https://www.malinc.se/noneuclidean/en/poincaretiling.php.

[2] Code Golf Stack Exchange. *Impress Donald Knuth by Counting Polyominoes on the Hyperbolic Plane.* Accessed: November 30, 2024. 2019. URL: https://codegolf.stackexchange.com/questions/200122/impress-donald-knuth-by-counting-polyominoes-on-the-hyperbolic-plane.

[3] Greg Malen, Érika Roldán, and Rosemberg Toalá-Enríquez. "Extremal $\{p, q\}$-animals". In: *Annals of Combinatorics* 27.1 (2023), pp. 169–209.