Existence and Stability of Standing Waves in Heterogeneous Reaction-Diffusion Equations

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Introduction

Motivated by problems in ecology, we investigate:

1) The survival of populations within a habitat patch and establish criteria for determining minimum habitat lengths.
2) The "standing wave problem," where habitat collapse occurs when the patch becomes too small, using reaction-diffusion equations and dynamical systems.
3) Pulse-like standing waves in an Allee-type growth model through numerical methods, aiming to determine precise length thresholds for population persistence.

The Model

In order to model these population dynamics, we use the following nonlinear RDE (Reaction-Diffusion Equation) in one spatial dimension:

$$u_t = Du_{xx} + f(u, H(x)), \qquad (*)$$

with an Allee growth term

$$f(u, H(x)) = -\beta u + \lambda H(x) u^2 - \gamma u^3$$

where

$$H(x) = \begin{cases} h^* \in \mathbb{R}, & \text{if } |x| \le \frac{L}{2} \\ 0, & \textit{otherwise} \end{cases}$$

with constant diffusion coefficient $D > 0$ and boundary conditions

$$\lim_{x \to \pm\infty} u(x,t) = 0.$$

This model accounts for the limited resources in a large population density and for under-crowding at lower population density. We investigate steady-state solutions (where $u_t = 0$) with parameters:

$$\beta \ge 0, \quad \gamma \ge 0, \quad \lambda = 1,$$
$$D = 1, \quad h^* = 1.$$

Hence, we obtain the following ODEs for being "inside" and "outside" the habitat:

$$\begin{cases} u_{xx} - \beta u + u^2 - \gamma u^3 = 0, & \text{if } |x| \le \frac{L}{2} \\ u_{xx} - \beta u - \gamma u^3 = 0, & \textit{otherwise} \end{cases}$$

With the use of reduction of order, we obtain two first-order systems:

Inside habitat:	**Outside habitat:**
$u_x = v$	$u_x = v$
$v_x = \beta u - \lambda h^* u^2 + \gamma u^3$	$v_x = \beta u + \gamma u^3$

From the inside habitat ODE, we establish a critical length L^*, where $\frac{L^*}{2}$ represents the minimum half-length required for population survival. When $L < L^*$, the population cannot persist.

Results

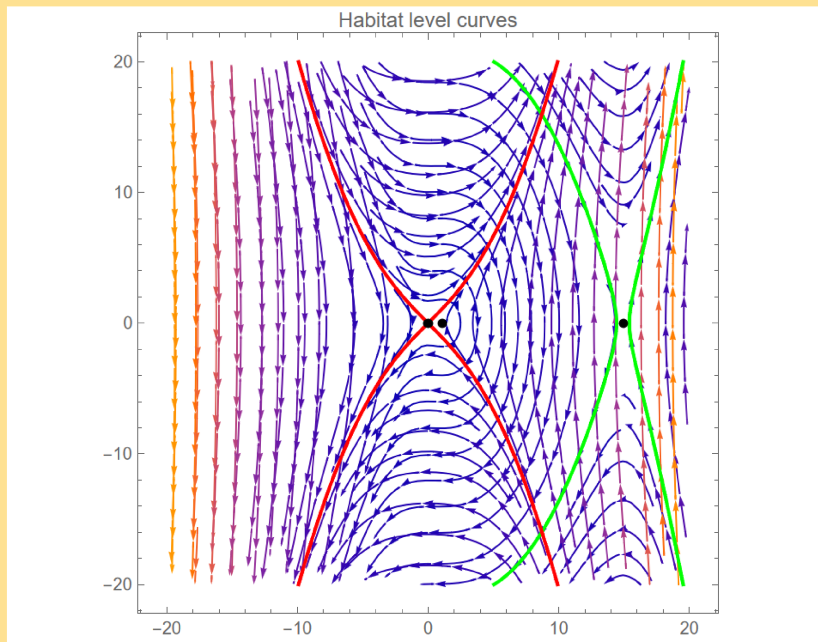
The Habitat

The inside and outside habitat ODEs are Hamiltonian, so we can obtain the following Hamiltonian functions and plot solution trajectories:

$$E(u,v) = \begin{cases} \frac{v^2}{2} - \frac{\beta u^2}{2} + \frac{u^3}{3} - \frac{\gamma u^4}{4} - k, & \sqrt[3]{3k} < u < \frac{1+\sqrt{1-4\gamma\beta}}{2\gamma} \\ \frac{v^2}{2} - \frac{\beta u^2}{2} - \frac{\gamma u^4}{4}, & 0 < u < \sqrt[3]{3k} \end{cases}$$

where we find that

$$0 < k < -\frac{\beta u^2}{2} + \frac{u^3}{3} - \frac{\gamma u^4}{4}, \; u = \frac{1+\sqrt{1-4\gamma\beta}}{2\gamma}$$
$$0 < \beta < \beta_{crit}$$
$$0 < \gamma < \gamma_{crit}$$



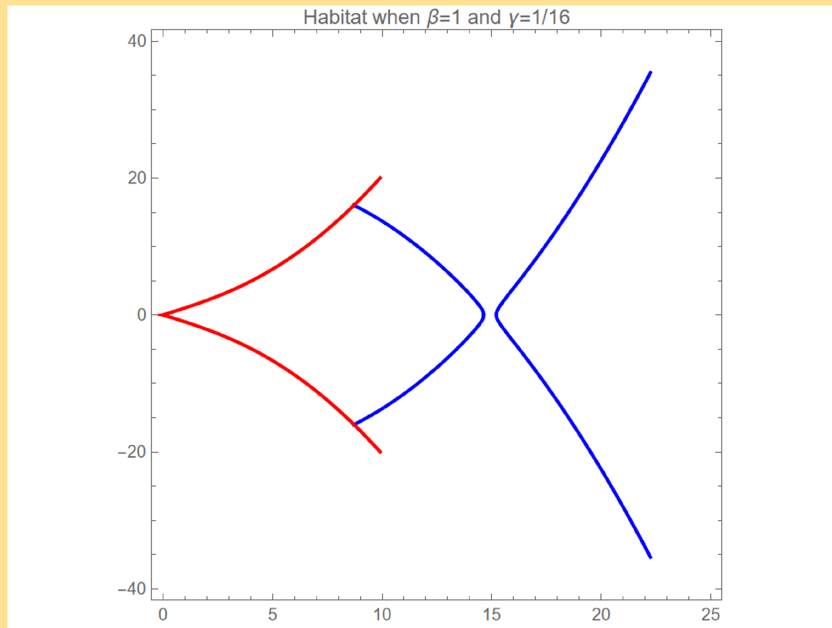
Red – Outside Habitat at k = 0, Blue – Inside Habitat at k = 220

Energy around Fixed Points

For the inside habitat function, we identify up to three fixed points. Using bifurcation theory, we find that stability is lost when $\gamma > \gamma_{crit}$ or $\beta > \beta_{crit}$. The system exhibits stability when exactly three fixed points exist, which are:

$$(u^*, v^*) = \begin{cases} (0,0) \\ (u_-, 0) := \left(\frac{1-\sqrt{1-4\gamma\beta}}{2\gamma}, 0\right) \\ (u_+, 0) := \left(\frac{1+\sqrt{1-4\gamma\beta}}{2\gamma}, 0\right) \end{cases}$$

We also observe that $E(0,0) > 0$, $E(u_+, 0) > 0$, and $E(u_-, 0) < 0$, which leads to the constraint $0 < \gamma < \frac{2}{9\beta}$.



Red – Outside Habitat at k = 0, Blue – Inside Habitat at k = 220.99

Critical Length of Habitat

The habitat collapses when its length falls below a critical value L^*. From the inside habitat Hamiltonian and using separation of variables with constant k (dependent on choice of u_0), we derive the length formula:

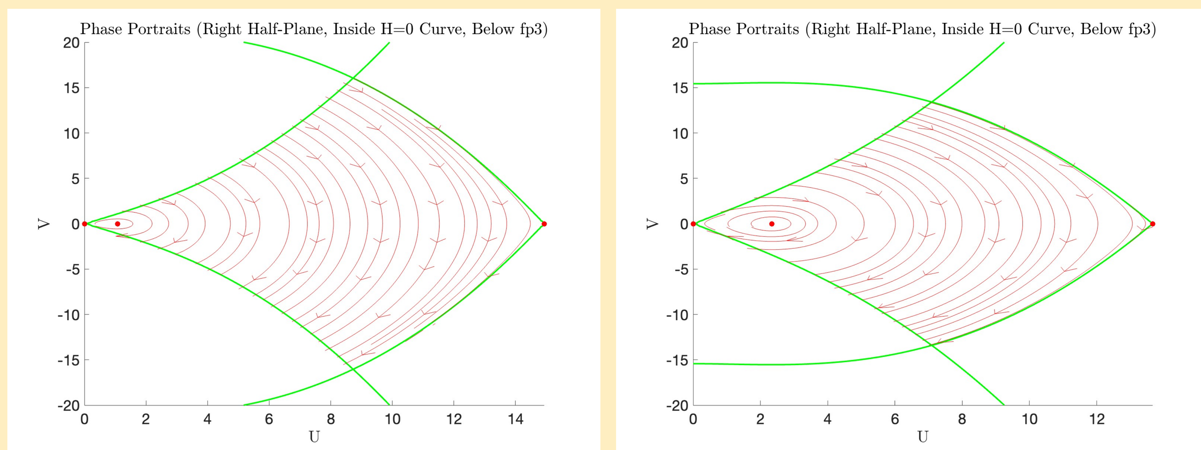
$$L(u_0) = 2\int_{u_0}^{u_1} \frac{1}{\sqrt{\beta\tau^2 - \frac{2}{3}\tau^3 + \frac{1}{2}\gamma\tau^4 + 2k(u_0)}} d\tau$$

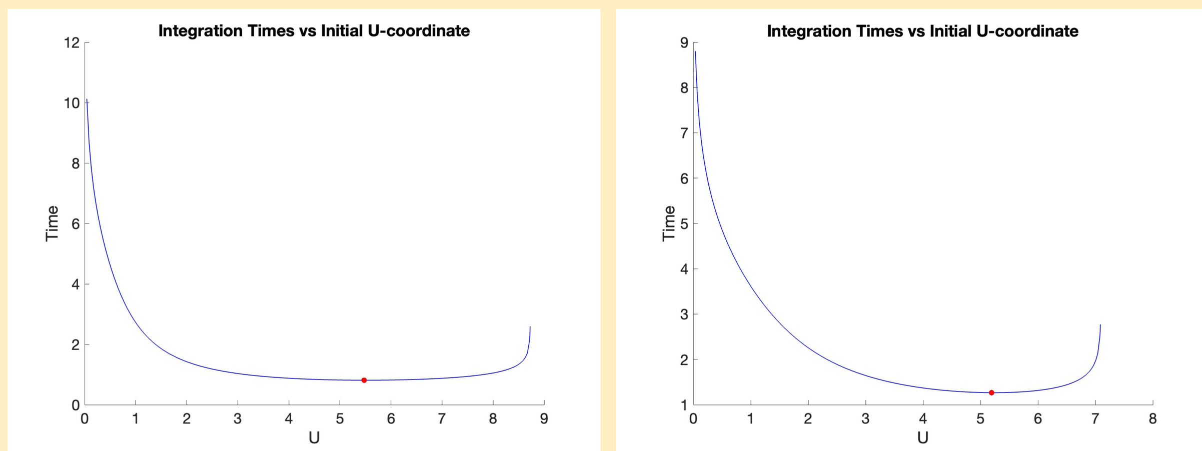
Theorem

Consider the RDE (*) with $u : \Omega \times [0,\infty) \to (0,\infty)$ and parameters $\beta > 0$, $h^* = \lambda = 1$, and γ satisfying $0 < \gamma < \frac{2}{9\beta}$. Let $L : (0, u_+) \to (0, \infty)$ be C^2. If:

1) $\lim_{u_0 \downarrow 0} L(u_0) = \infty$
2) $\lim_{u_1 \uparrow u_+} L(u_1) = \infty$
3) $L''(u_0) > 0$ *for all* $u_0 \in (0, u_+)$

then there exists $L^* > 0$ such that if $L < L^*$, the habitat collapses: $\lim_{t\to\infty} \sup_{x\in\Omega} u(x,t) = 0$.





Numerical Solutions to the Standing Wave Problem

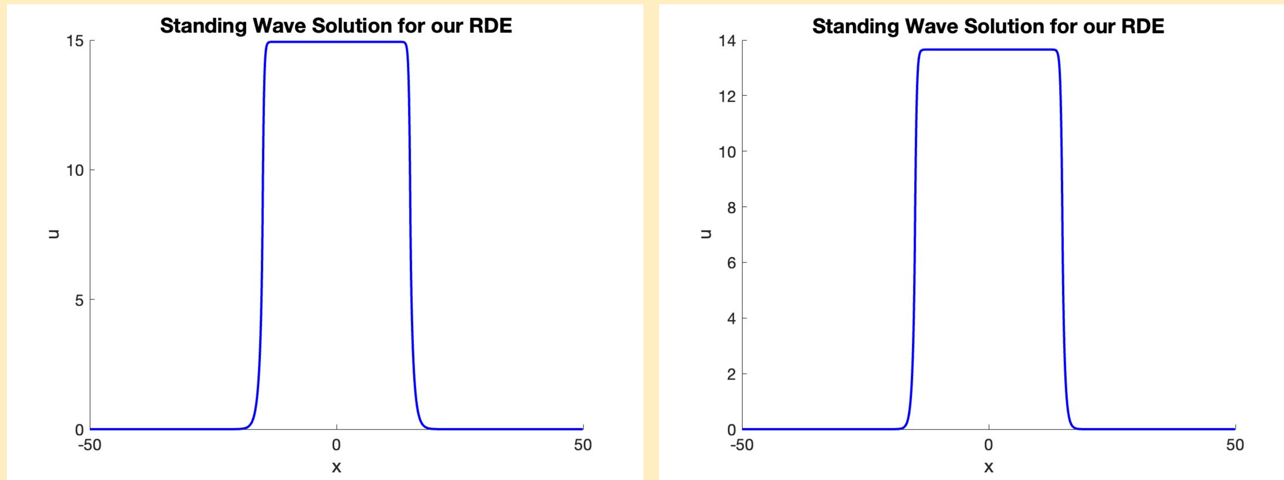
The numerical method we used to compute the standing wave solution is the finite difference method. Computing the finite difference approximation,

$$u_{i,j} - \frac{\Delta t}{2(\Delta x)^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) = u_{i,j-1} + \Delta t \cdot f(u_{i,j-1}, H(x_i)).$$

From there we created a matrix M so that

$$Mu_{i,j} = u_{i,j-1} + \Delta t \cdot f(u_{i,j-1}, H(x_i)).$$

Next we took the inverse of M on each side of the equation.



Plots of the solutions for $\beta = 1$ and $\beta = 2$ where $\gamma = 1/16$, $\lambda = 1$ and $h^* = 1$ for both

Conclusions/Future Work

In future work, we aim to show that the $L(u_0)$ function implies two standing waves in the system—one stable and one unstable—and further investigate the unstable wave. We will also explore the traveling wave problem by adopting a moving-frame coordinate system, redefining the habitat function as $H(x - ct)$ with speed c. Using a similar approach, we will determine a critical speed c^* at which a population can remain stable while moving.

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References

- Berestycki, H., et al. "Can a Species Keep Pace with a Shifting Climate?" Bulletin of Mathematical Biology, vol. 71, no. 2, 9 Dec. 2008, pp. 399–429, doi:10.1007/s11538-008-9367-5.
- Hasan, Cris R., et al. "Rate-Induced Tipping in Heterogeneous Reaction-Diffusion Systems: An Invariant Manifold Framework and Geographically Shifting Ecosystems." SIAM Journal on Applied Dynamical Systems, vol. 22, no. 4, 30 Oct. 2023, pp. 2991–3024, doi:10.1137/22m1536625.