Exploring the Grothendieck Group via Character Varieties

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$\mathrm{SL}_2\mathbb{C}$ Character Variety: The Recipe

1. A finitely presentable group $\Gamma = \langle \gamma_1, \ldots, \gamma_m | R_1, \ldots, R_l \rangle$
2. The collection of group homomorphisms $\mathrm{Hom}(\Gamma, \mathrm{SL}_2\mathbb{C})$.
3. The collection $\mathrm{Hom}(\Gamma, \mathrm{SL}_2\mathbb{C})/\sim$ of conjugacy classes with closed orbit.

Lemma

The space $Hom(\Gamma, \mathrm{SL}_2\mathbb{C})$ is in bijective correspondence with the collection $\{(A_1, \ldots, A_m) \in \mathrm{SL}_2\mathbb{C}^m | R_i(A_1, \ldots, A_m) - I_2 = 0, \text{ for all } 1 \leq i \leq l\}$.

For this project, we are concerned mainly with two-generator one-relation groups. That is, $\Gamma = \langle a, b | R \rangle$. For this, we have the following alternative construction.

Lemma

The $SL_2\mathbb{C}$ character variety of $\Gamma = \langle a, b | R \rangle$ is in bijective correspondence with the collection of maximum ideals of the coordinate ring $\mathbb{C}[x, y, z]/\langle \mathrm{tr}(R) - 2, \mathrm{tr}(AR) - \mathrm{tr}(A), \mathrm{tr}(BR) - \mathrm{tr}(B) \rangle$ where $x = \mathrm{tr}(A)$, $y = \mathrm{tr}(B)$ and $z = \mathrm{tr}(AB)$.

A Simple Example

Example

Let $\Gamma = \mathbb{Z}_2$. Then the $\mathrm{SL}_2\mathbb{C}$ character variety of Γ is given by the collection of matrices $\{A \in \mathrm{SL}_2\mathbb{C} | A^2 = I_2\}/\sim$. The only such matrices are $\pm I_2$. Then through the characteristic equation, these matrices are in bijection with their trace, and so we have that the character variety is $\{-2, 2\}$ by the first lemma.

Example

Let $\Gamma = \mathbb{Z}_2$. Then the $\mathrm{SL}_2\mathbb{C}$ character variety of Γ is the collection of maximal ideals of the coordinate ring $\mathbb{C}[x]/\langle x^2 - 4\rangle$. So the character variety is, by the previous lemma, $\{-2, 2\}$ as expected.

A Neat Example: $\mathbb{Z} \oplus \mathbb{Z}$

Example

Let $\Gamma = \langle a, b | aba^{-1}b^{-1} \rangle = \mathbb{Z} \oplus \mathbb{Z}$. The corresponding space $\mathrm{Hom}(\Gamma, \mathrm{SL}_2\mathbb{C})$ is in bijective correspondence with $\{(A, B) | AB = BA\}$ with closed orbits. It is known that such matrices are simulatneously upper triagonalizable, so up to conjugacy, this collection is in bijective correspondence with the collection of pairs $\left(\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \right)$.

This suggests the correspondence

$$\left(\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \right) \mapsto (\lambda, \tfrac{1}{\lambda}) \in \mathbb{C}^\times \times \mathbb{C}^\times$$

We only need now to identitfy $(\lambda, \frac{1}{\lambda}) \sim (\frac{1}{\lambda}, \lambda)$. This yields the character variety as $\mathbb{C}^\times \times \mathbb{C}^\times / \mathbb{Z}_2$.

Semester Results

1. Complete classification of character varieties of cyclic groups.
2. Complete classification of character varieties of the form $\Gamma = \langle a, b | a^n \rangle$
3. Complete classification of character varieties on 2-generator one-relation groups up to word length 3.

Character Variety of $\mathbb{Z}_n$

Let $\Gamma = \mathbb{Z}_n$. We answer the following questions.

1. Is there an explicit formula for the polynomials defining the character variety?
2. How many points are in the character variety?

Polynomials Defining Character Variety for Z_n

We know that the polynomials generating the character variety for $\mathbb{Z}_n$ are given recursively by

1. $f_1(t) = t$
2. $f_2(t) = t^2 - 2$
3. $f_n(t) = tf_{n-1}(t) - f_{n-2}(t)$

This is fine, but computationally expensive for large n.

Explicit Formula for $f_n(t)$

f(t)
t
$t^2 - 2$
$t^3 - 3t$
$t^4 - 4t^2 + 2$
$t^5 - 5t^3 + 5t$
$t^6 - 6t^4 + 9t^2 - 2$
$t^7 - 7t^5 + 14t^3 - 7t$
$t^8 - 8t^6 + 20t^4 - 16t^2 + 2$
$t^9 - 9t^7 + 27t^5 - 30t^3 + 9t$
$t^{10} - 10t^8 + 35t^6 - 50t^4 + 25t^2 - 2$

Figure: First ten polynomials of $f_n(t)$

Explicit Formula for $f_n(t)$

f_n(t)	T_n(t)
t	t
$t^2 - 2$	$2t^2 - 1$
$t^3 - 3t$	$4t^3 - 3t$
$t^4 - 4t^2 + 2$	$8t^4 - 8t^2 + 1$
$t^5 - 5t^3 + 5t$	$16t^5 - 20t^3 + 5t$
$t^6 - 6t^4 + 9t^2 - 2$	$32t^6 - 48t^4 + 18t^2 - 1$
$t^7 - 7t^5 + 14t^3 - 7t$	$64t^7 - 112t^5 + 56t^3 - 7t$
$t^8 - 8t^6 + 20t^4 - 16t^2 + 2$	$128t^8 - 256t^6 + 160t^4 - 32t^2 + 1$
$t^9 - 9t^7 + 27t^5 - 30t^3 + 9t$	$256t^9 - 576t^7 + 432t^5 - 120t^3 + 9t$
$t^{10} - 10t^8 + 35t^6 - 50t^4 + 25t^2 - 2$	$512t^{10} - 1280t^8 + 1120t^6 - 400t^4 + 50t^2 - 1$

Figure: First ten polynomials of $f_n(t)$ next to first ten Chebyshev Polynomials

Where here $T_n(t)$ represents the nth Chebyshev polynomial of the first kind.

Explicit Formula

Lemma

Let $\Gamma = \mathbb{Z}_n$. Then the $SL_2\mathbb{C}$ character variety of $\mathbb{Z}_n$ is defined by the polynomial $f_n(t) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k [\binom{n-k}{k} + \binom{n-k-1}{n-2k}] t^{n-2k}$

Observation: The nth Chebyshev polynomial of the first kind is given explicitly by $T_n(t) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k [\binom{n-k}{k} + \binom{n-k-1}{n-2k}] 2^{n-2k-1} t^{n-2k}$

Size of Character Variety of $\mathbb{Z}_n$

Lemma

Let $\Gamma = \mathbb{Z}_n$. Then the $SL_2\mathbb{C}$ character variety of $\mathbb{Z}_n$ is in bijective correspondence with the collection $\{\mathrm{tr}(A) | A \in \mathrm{SL}_2\mathbb{C}, A^n = 1\}$

Corollary

The number of points in the $SL_2\mathbb{C}$ character variety of $\mathbb{Z}_n$ is $s(n) = \lfloor \frac{n}{2} \rfloor + 1$.

Corollary

The $SL_2\mathbb{C}$ character variety of $\mathbb{Z}_n$ is given by $\{2\cos(\frac{2\pi k}{n}) | k \in \mathbb{Z}\}$.

Proof that $s(n) = \lfloor \frac{n}{2} \rfloor + 1$

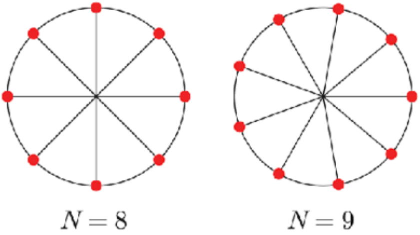

Figure: Roots of Unity

Proof.

Recall that $Hom(\mathbb{Z}_n, SL_2\mathbb{C})/\sim \cong \{A \in SL_2\mathbb{C} | A^n = 1\}/\sim \cong \left\{ \begin{pmatrix} e^{i\frac{2\pi k}{n}} & 0 \\ 0 & e^{-i\frac{2\pi k}{n}} \end{pmatrix} \middle| k \in \mathbb{Z} \right\} /\sim$. These are precisely the n roots of unity. So $s(n)$ is just the number of distinct traces of matrices in this collection. Hence $s(n) = \lfloor \frac{n}{2} \rfloor + 1$. □

Two Generators, Cyclic Relation

We now study the case where $\Gamma = \langle a, b | a^n \rangle$.
We have three variables that define our polynomials $tr(A)$, $tr(B)$ and $tr(AB)$.
And we get the trace relations defining the character variety,

1. $tr(A^n) - tr(I) = 0$
2. $tr(A^{n+1}) - tr(A) = 0$
3. $tr(BA^n) - tr(B) = 0$

Polynomials Defining the $\Gamma = \langle a, b | a^n \rangle$ Character Variety

Let $g_n(x, y, z) = tr(BA^n) - tr(BA)$. Then using the trace relations and defining $tr(A) = x$, $tr(B) = y$, and $tr(AB) = z$ we obtain the following functions which generate our character varieties

1. $g_1(x, y, z) = 0$
2. $g_2(x, y, z) = xy - z - y$
3. $g_n(x, y, z) = xg_{n-1}(x, y, z) - g_{n-2}(x, y, z)$

Character Variety of $\langle a, b | a^n \rangle$

Lemma

Let $\Gamma = \langle a, b | a^n \rangle$*, then the* $SL_2\mathbb{C}$ *character variety of* Γ *is of the form* $\mathbb{C} \sqcup \bigsqcup_{i=1}^{N} \mathbb{C}^2$ *where* $N = \lfloor \frac{n}{2} \rfloor + 1$ *when* n *is odd, and* $\mathbb{C} \sqcup \mathbb{C} \sqcup \bigsqcup_{i=1}^{N} \mathbb{C}^2$ *where* $N = \lfloor \frac{n}{2} \rfloor + 1$ *when* n *is even.*

Intuition: A generates a cyclic group of order n and is of fixed values while B is always able to act freely and acts separately from AB when A is not the identity.

Explicit Example

Example

Let $\Gamma = \langle a, b | a^3 \rangle$. Then the polynomials defining the character variety are

1. $f_1(x, y, z) = x^3 - 3x - 2$
2. $f_2(x, y, z) = x^4 - 4x^2 - x + 2$
3. $f_3(x, y, z) = zx^2 - z - xy - y$

The common zeros of these are $(-1, y, z)$ and $(2, y, y)$ for any $y, z \in \mathbb{C}$. Thus the character variety is in bijective correspondence with $\mathbb{C} \sqcup \mathbb{C}^2$.

Classification of Low Complexity Groups

Word Length 2	
Word	**Variety**
aa	C+C
bb	C+C
a^{-1}a^{-1}	C+C
b^{-1}b^{-1}	C+C
Others	C

Word Length 3	
Word	**Variety**
aaa	C+C^2
bbb	C+C^2
a^{-3}	C+C^2
b^{-3}	C+C^2
Others	C

Future Project Goals

1. Continue classifying character varieties of finite two generator groups in $SL_2\mathbb{C}$.
2. Begin classifying the character varieties of finitely presentable three generator groups in $SL_2\mathbb{C}$ or two generator groups with two words.