Exploring the Groethendieck group via character varieties

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April 26, 2024

Introduction

The purpose of this project was to explore $SL_2\mathbb{C}$ character varieties. Loosely speaking, these varieties arise from looking at matrix representations of homomorphisms into $SL_2\mathbb{C}$ from some group. If Γ is some group, one could ask what is the $SL_2\mathbb{C}$ character variety of Γ? The purpose of this project was precisely to answer this question for low complexity groups. We completely answered the question in the case that Γ is cyclic, and we provided answers in some other cases.

Definition (Character Variety)

Let $\Gamma = \langle \gamma_1, \ldots, \gamma_n | R_1, \ldots, R_m \rangle$ be some finitely presentable group. The $SL_2\mathbb{C}$ character variety is the quotient Hom(Γ,SL$_2$C)//$\sim$ of conjugacy classes with closed orbits.

Lemma (As collection of matrices)

The character variety can be represented as closed equivalence classes of homomorphisms from Γ into $SL_2(\mathbb{C})$, that is $Hom(\Gamma, SL_2C)//{\sim} \cong \{A_1, ..., A_n \in SL_2(\mathbb{C}) | R_1 = I, ..., R_m = I\}$. Where each word R_i corresponds to a word in the generators and requires that the corresponding word in the matrices $A_1, ..., A_n$ equals the identity matrix I. This perspective adds a concrete way to study the structure of Γ using both algebra and geometry.

Lemma (As polynomials)

The character variety can also be represented as polynomials using the Cayley-Hamilton theorem which states that $A^2 - tr(A)A + I = 0$ and, since traces are invariant under conjugation, we can use these to describe the character variety. The relations $R_1, ..., R_m$ translate into polynomial equations in terms of traces, which are $tr(R_i) - tr(I) = 0$. By using both the Cayley-Hamilton theorem and the polynomials induced by the relations of Γ, the character variety can be described as a solution set of polynomial equations in the traces of matrices corresponding to the group generators.

A nice example

Commutator Word

Let $\Gamma = \langle a, b | aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. The $SL_2\mathbb{C}$ character variety of Γ is $\mathbb{C}^* \times \mathbb{C}^*/\mathbb{Z}_2$.

Proof.

Recall that the character variety of $\mathbb{Z} \oplus \mathbb{Z}$ is in one to one correspondence with the collection of matrices (A, B) where $AB = BA$. It is known that such matrices are simultaneously upper-triagonalizable, so up to conjugacy, the character variety is the set of all pairs

$$\left(\begin{pmatrix} \lambda & x_1 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \begin{pmatrix} \lambda & x_2 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \right)$$

This suggests the correspondence

$$\left(\begin{pmatrix} \lambda & x_1 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \begin{pmatrix} \lambda & x_2 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \right) \mapsto (\lambda, \frac{1}{\lambda}) \in \mathbb{C}^\times \times \mathbb{C}^\times$$

up to the identification of $(\lambda, \frac{1}{\lambda}) \sim (\frac{1}{\lambda}, \lambda)$ giving the desired result. □

Commutator Word

Alternatively, we can find the character variety of $\mathbb{Z} \oplus \mathbb{Z}$ by finding the simultaneous zeros of the polynomials. The real part of this variety is depicted below.

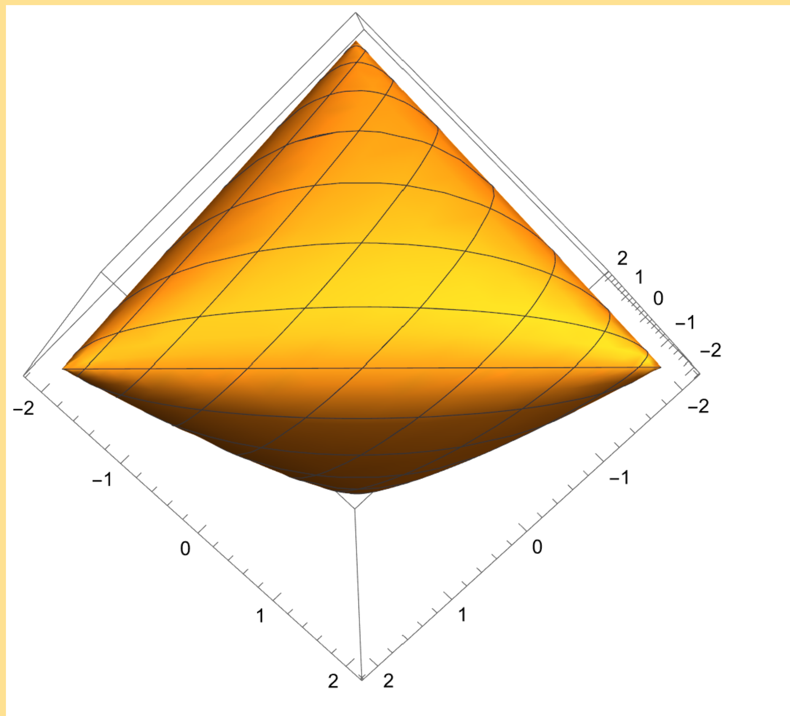

Figure: Real part of $\mathbb{Z} \oplus \mathbb{Z}$ character variety

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$SL_2\mathbb{C}$ Character Variety of $\mathbb{Z}_n$

Let $\Gamma = \mathbb{Z}_n$. With some work, we were able to conclude the following about the $SL_2\mathbb{C}$ character variety of Γ.

1. The number of points in the character variety is always finite. In particular, the number is given by $\lfloor \frac{n}{2} \rfloor + 1$
2. The recursively defined polynomials defining the character variety are given explictly by $f_n(t) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k [\binom{n-k}{k} + \binom{n-k-1}{n-2k}] t^{n-2k}$
3. The character variety is given by $\{\frac{2\pi k}{n} | k \in \mathbb{Z}\}$.

Two generators one relation

Let $\Gamma = \langle a, b | a^n \rangle$. Then The $SL_2\mathbb{C}$ character variety of Γ is given by $\mathbb{C} \sqcup \bigsqcup_{i=1}^{N} \mathbb{C}^2$ where $N = \lfloor \frac{n}{2} \rfloor + 1$ when n is odd, and $\mathbb{C} \sqcup \mathbb{C} \sqcup \bigsqcup_{i=1}^{N} \mathbb{C}^2$ where $N = \lfloor \frac{n}{2} \rfloor + 1$ when n is even.

Example

Let $\Gamma = \langle a, b | a^3 \rangle$. Then the polynomials defining the character variety are $f_1(x, y, z) = x^3 - 3x - 2, f_2(x, y, z) = x^4 - 4x^2 - x + 2$ and $f_3(x, y, z) = zx^2 - z - xy - y$. The common zeros of these are $(-1, y, z)$ and $(2, y, y)$ for any $y, z \in \mathbb{C}$. Thus the character variety is in bijective correspondence with $\mathbb{C} \sqcup \mathbb{C}^2$.

Classification up to Word Length 4

Word Length 2	
Word	Variety
aa	C+C
bb	C+C
a^{-1}a^{-1}	C+C
b^{-1}b^{-1}	C+C
Others	C

Word Length 3	
Word	Variety
aaa	C+C^2
bbb	C+C^2
a^{-3}	C+C^2
b^{-3}	C+C^2
Others	C

Acknowledgments

Special thanks to Dr. Sean Lawton for mentoring and overseeing this project, as well as our graduate mentor Gabe Lumpkin.

References

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