Knots and Links in Thickened Surfaces

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Introduction

In this project, we studied hyperbolic links in thickened surfaces. If the link only has one component, then it is called a *knot*. A link is *alternating* if it admits a diagram with an orientation such that as we travel along the path of the link, the strands have a pattern of going over a crossing then under the next. Links have traditionally been studied in the 3-sphere or $\mathbb{R}^3$ but they can also be generalized to other spaces such as thickened surfaces. In this research, we explored generalized alternating links on the genus 2 torus that correspond to checkerboard tilings of the hyperbolic plane by n and $m-$ sided regular polygons.

Some Definitions

Definition (Link)

A **link** L of m components is a subset of 3-space, that consists of m disjoint simple closed curves. A link of one component is a knot.

Definition (Euler Characteristic)

The **Euler characteristic** of a surface S is given by $\chi(S) = V - E + F$, where V is the number of vertices in a polygonal decomposition of S, E is the number of edges, and F is the number of faces.

Definition (Alternating Link)

An alternating diagram is a diagram of a knot or link that has an orientation such that, when following the knot in the direction of the orientation, the crossings alternate between over and under. An **alternating knot** or **link** is a knot or link that has an alternating diagram.

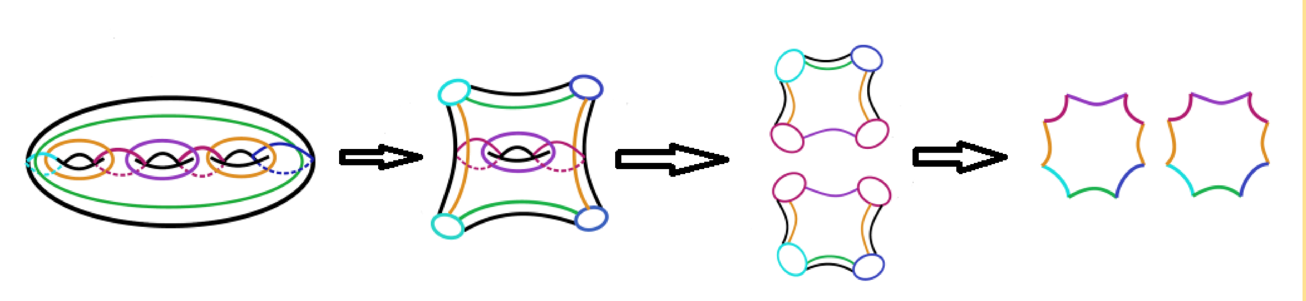
Definition (Hyperbolic Plane)

The Poincaré Disk Model of the hyperbolic plane $\mathbb{H}^2$ is defined as follows. The points are the points in the interior of the unit disk D. The geodesics are the pieces of circles that are perpendicular to D and diameters of D. The hyperbolic distance from a to b, denoted dH(a, b) is given by

$$d_H(a, b) = \int_\gamma \frac{2ds}{1 - r^2}$$

Link to Tiling

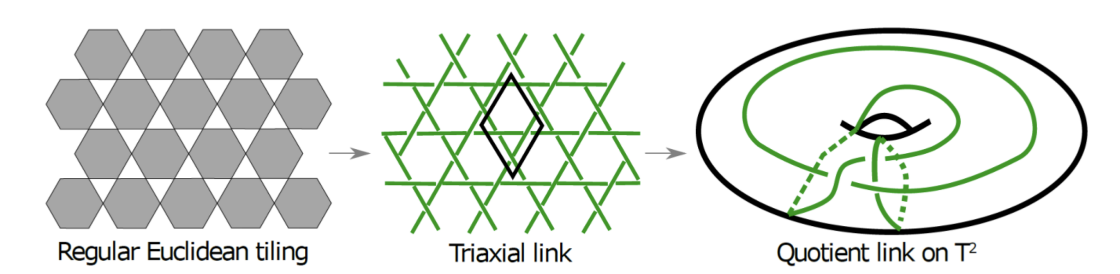
Given a link on a genus-3 surface, we can deconstruct the surface into a tiling.



Constructing Generalized Alternating Links

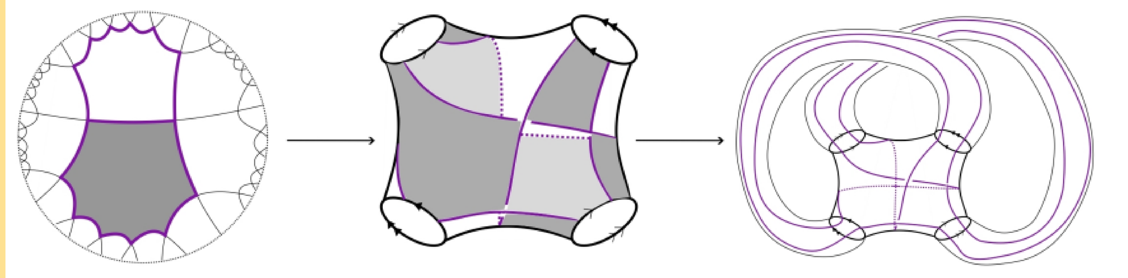
Euclidean Tiling

A Euclidean tiling covers the plane in polygons edge to edge without overlapping. The edges of the polygons become the strands of the link, and the vertices become the crossings. Then the link is glued onto the torus.

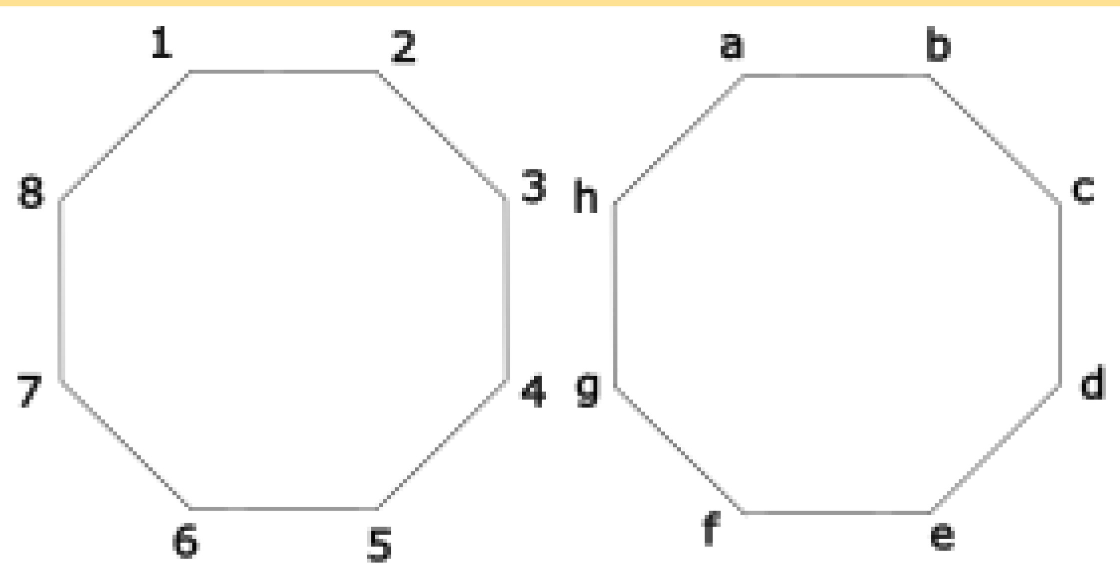


Hyperbolic Tiling

A Hyperbolic tiling uses tiles on the *Poincaré disk* where we used the tiling and manipulated it into our torus with the edges of our tiles being the string. What makes the new technique useful is that instead of focusing on the edges of the tiling we decided to focus on the vertices that form the crossings of our links.



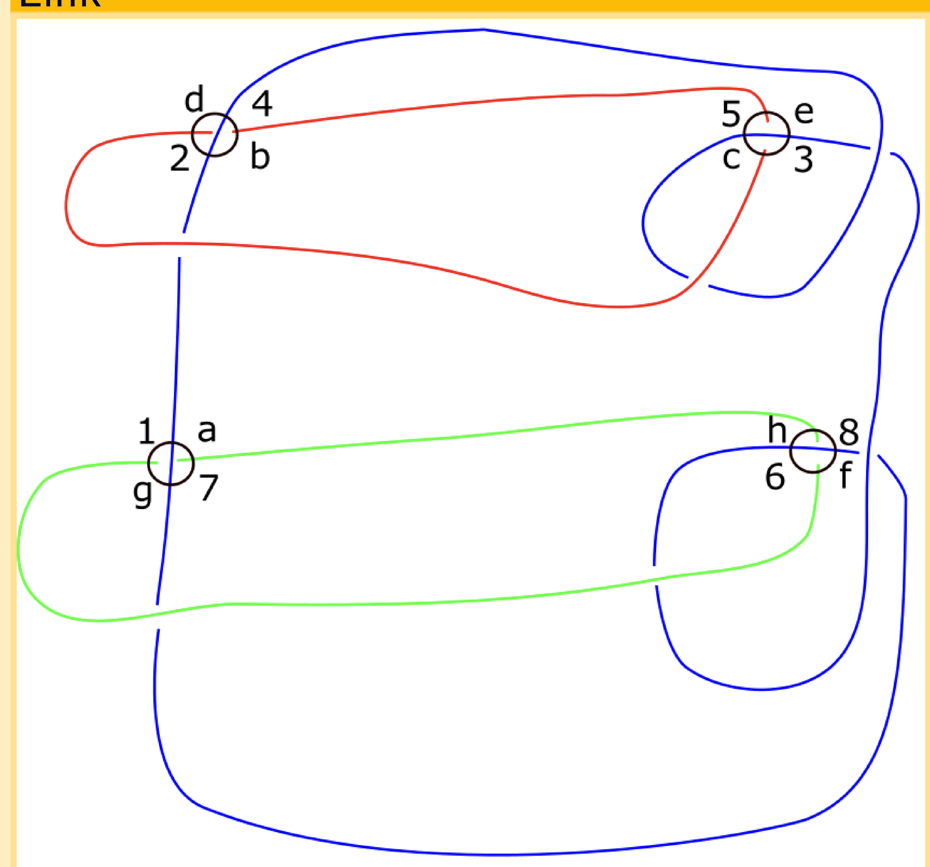
Building a knot using the vertices



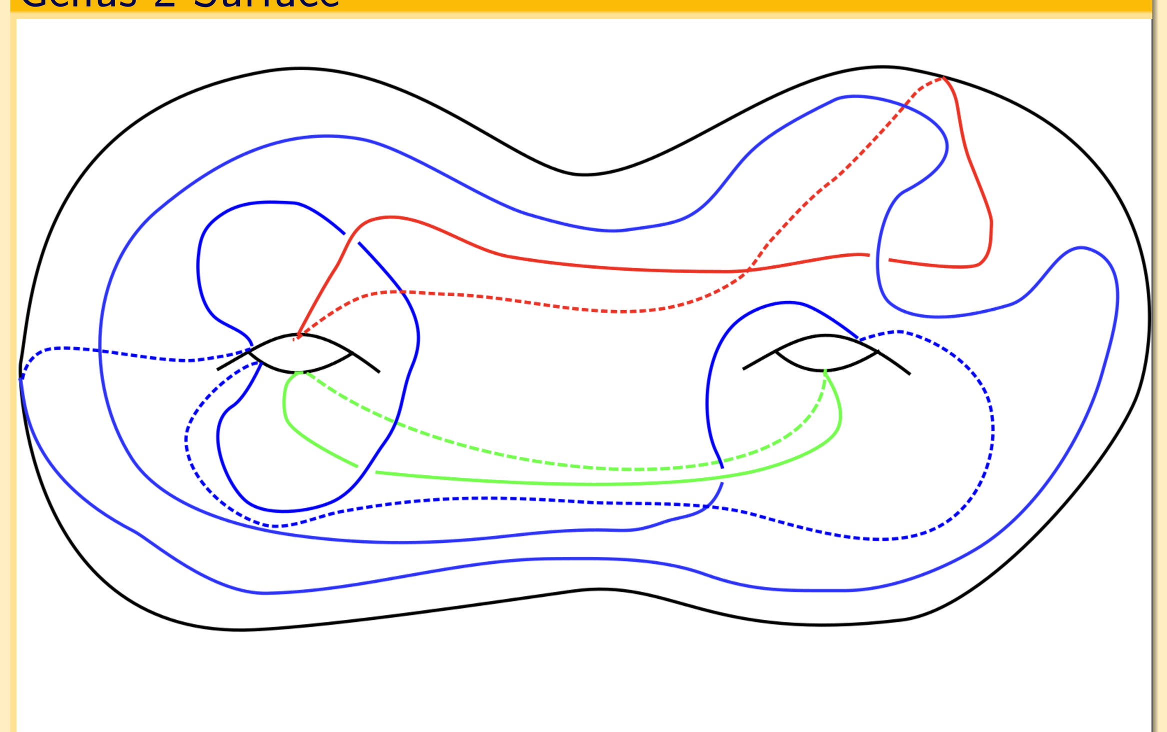
1	2	a	b		1	a
2	3	d	e	g		7
3	4	c	d		2	b
4	5	b	c	d		4
5	6	e	f		3	e
6	7	h	a	c		5
7	8	g	h		6	f
8	1	f	g	h		8

Graphics

Link

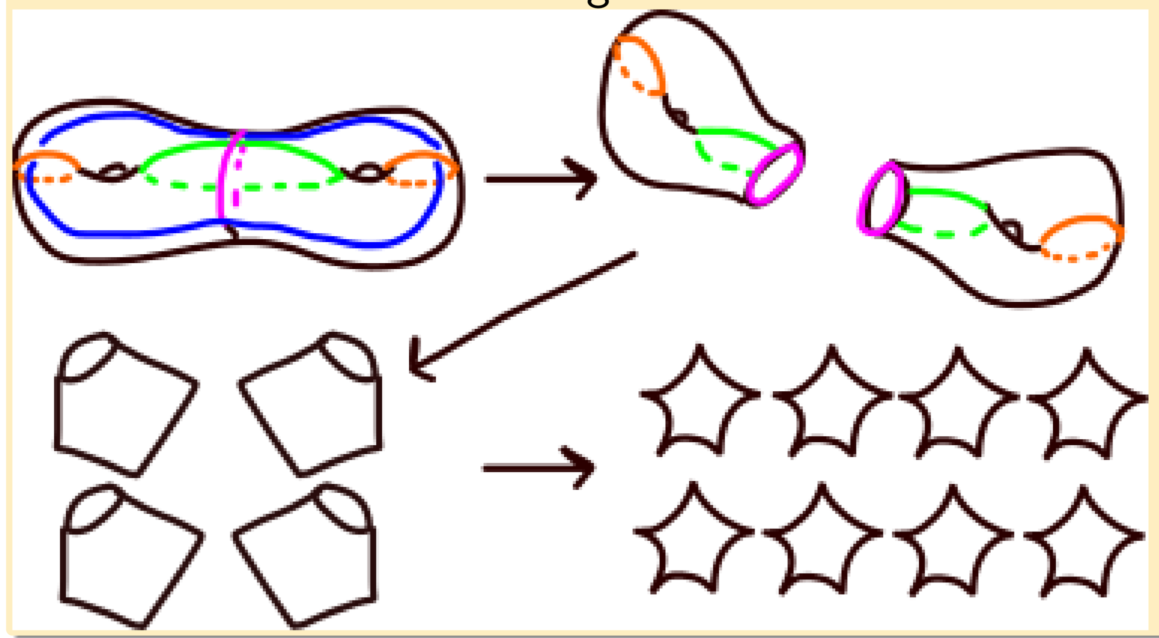


Genus-2 Surface



Link to Tiling of Pentagons

The following link on a genus-2 surface can be deconstructed into a tiling by 8 pentagons. This is one of many potential links that can come from this tiling.



Conclusions/Future Work

In this project, we developed a new method for producing all possible links from a given set of tiles, made a method to build the links in a visually easy way, learned the fundamentals of knot theory, and found a few link diagrams on surfaces that relate to tilings of hyperbolic space. From here we would like to continue using the new method to see all the possible iterations of link diagrams for different tilings, examine them with different invariants, and check how they can be placed on the genus two torus.

Remaining Questions

- How many distinct alternating links can be obtained from any one tiling?
- What other alternating links can be obtained from 2 octogons?
- Which invariants should be used to distinguish generalized alternating links from one another?

Acknowledgments

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References

1 "An Introduction to Knot Theory" by Raymond Lickorish
2 "Hyperbolic Knot Theory" by Jessica S. Purcell
3 "The Tiling Book" by Colin Adams