Universal Cycles in Higher Dimensions

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Universal Words

A *universal word* for $\mathcal{A}^n$ is a word that covers each word in $\mathcal{A}^n$ exactly once.

00110 is a universal word for $\{0, 1\}^2$, covering 00, 01, 10, 11.

We say universal word is *partial* if it contains any $\diamondsuit$ characters, which can represent any symbol in $\mathcal{A}$.

$\diamondsuit\diamondsuit$0111 is a universal partial word for $\{0, 1\}^3$, covering 000, 001, 010, 011, 100, 101, 110, and 111. [2]

De Bruijn Cycles and Upcycles

A *De Bruijn* cycle for $\mathcal{A}^n$ is a **cyclic** sequence that contains each word in $\mathcal{A}^n$ exactly once. [1]

0011 is a De Bruijn Cycle for $\{0,1\}^2$

001122102 is a De Bruijn Cycle for $\{0,1,2\}^2$

A *universal partial cycle* (upcycle) for $\mathcal{A}^n$ is a **cyclic** word that covers each word in $\mathcal{A}^n$ exactly once using a wildcard character $\diamondsuit$ that can represent any element of $\mathcal{A}$. [2]

001$\diamondsuit$110$\diamondsuit$ is a universal partial cycle [2] for $\{0,1\}^4$.

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Binary alphabet, sub-matrix dimensions 2×2: $(4,4,2,2)_2$

Sixteen possible words:

$\begin{matrix}0&1\\0&0\end{matrix}$ $\begin{matrix}0&0\\0&1\end{matrix}$ $\begin{matrix}0&0\\1&0\end{matrix}$ $\begin{matrix}1&0\\0&0\end{matrix}$ $\begin{matrix}1&1\\0&0\end{matrix}$ $\begin{matrix}0&1\\0&1\end{matrix}$ $\begin{matrix}0&0\\1&1\end{matrix}$ $\begin{matrix}1&0\\1&0\end{matrix}$

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Universal Matrix:

$$\begin{matrix}1&1&1&0&1&0&0&1&0\\0&1&1&0&0&0&0&1&1\\1&0&0&1&1&0&1&1&0\end{matrix}$$

De Bruijn Torus:

$$\begin{matrix}0&1&0&0\\0&1&1&1\\1&1&1&0\\0&0&1&0\end{matrix}$$

Before Last Semester

Non-Cyclic Object Count

	1 Dimensional	2 Dimensional
$\mathcal{A}$	∞	∞
$\mathcal{A} \cup \{\Diamond\}$	∞	?

Cyclic Object Count

	1 Dimensional	2 Dimensional
$\mathcal{A}$	∞	∞
$\mathcal{A} \cup \{\Diamond\}$	∞	?

Research Questions

1. How many “universal partial matrices” are there?
2. How many “universal partial tori” (uptori) are there?
3. What constructions produce universal partial tori?

Now

Non-Cyclic Object Count

	1 Dimensional	2 Dimensional
$\mathcal{A}$	∞	∞
$\mathcal{A} \cup \{\Diamond\}$	∞	∞

Cyclic Object Count

	1 Dimensional	2 Dimensional
$\mathcal{A}$	∞	∞
$\mathcal{A} \cup \{\Diamond\}$	∞	∞

Non-cyclic Universal Matrices

Universal matrices can be created from De Bruijn Tori by translating the leftmost $\ell - 1$ column(s) and the topmost $w - 1$ row(s) near the right and bottom:

$$\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 & 1 \end{array}$$

We have found other universal arrays that cannot be made cyclic. For example:

$$\begin{array}{ccccc} 0 & 0 & 1 & 1 & 0 \\ 0 & \diamond & 1 & \diamond & 0 \\ 1 & 0 & 0 & 1 & 1 \end{array}$$

Constructed Universal Matrices

Given a universal partial word w for $\mathcal{A}^n$, we construct a $p \times |w|$ matrix μ whose first row is w and whose subsequent $p-1$ rows are $|w|$ $\diamond$'s:

$$\mu(w) = \begin{bmatrix} w \\ \diamond^{|w|} \\ \vdots \\ \diamond^{|w|} \end{bmatrix}$$

For example, choosing the universal word 0120221100 for $\{0,1,2\}^2$ and $p = 2$ produces the matrix

$$\mu = \begin{bmatrix} 0 & 1 & 2 & 0 & 2 & 2 & 1 & 1 & 0 & 0 \\ \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond & \diamond \end{bmatrix}$$

The matrix μ is a universal partial matrix for $\mathcal{A}^{p \times n}$.

Computationally Identified Universal Partial Tori

There are universal partial tori. Consider $(4,3,2,2)_2^{\diamondsuit=1}$:

$$\begin{matrix} \diamondsuit & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}$$

Larger sub-matrix sizes are computationally challenging, though we've made some progress on speeding up a program to search for them.

Constructions of De Bruijn Tori and Uptori

There are methods to construct De Bruijn tori (without wildcards), using De Bruijn cycles and related objects (alternating De Bruijn cycles and De Bruijn families) [3]

To produce an uptorus that covers sub-matrices with $m+1$ rows and n columns, begin by constructing an upcycle u over alphabet $\mathcal{A}$ with word length n. Next, construct a De Bruijn cycle d for $\{0...|u|-1\}^m$.

The uptorus is a stack of forwards rotated copies of u. The first row is u. For $0 < j < |d|$, The j^{th} row of the uptorus is the $(j-1)^{\text{th}}$ row rotated forwards by d_{j-1} characters.

Essentially, we treat the De Bruijn cycle as a cumulative rotation index for the upcycle.

Example of Construction and Proof

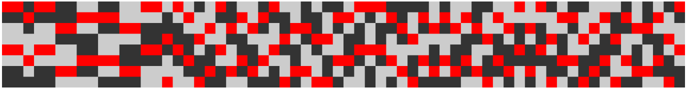
We are well on our way to proving that this construction produces infinitely many uptori from any given upcycle.

Consider upcycle $u = 001\diamondsuit 110\diamondsuit$

and De Bruijn Cycle $d =$

$$01702030405060711272231373341424744515253575561626364677766543210$$

We then construct an uptorus; the transpose is illustrated below:



A *universal partial family* for $\mathcal{A}^\ell$ is a set $\mathcal{S}$ of cyclic strings on $\mathcal{A} \cup \{\diamond\}$ such that every element of $\mathcal{S}$ has the same length, and every word in $\mathcal{A}^\ell$ is covered by exactly one element of $\mathcal{S}$, and is only covered once in that element.

Family $\mathcal{F}$ is a universal partial family for $0, 1, 2, 3^4$:

$$\mathcal{F} = \{ 001\diamond110\diamond003\diamond112\diamond021\diamond130\diamond023\diamond132\diamond, \\ 20103100201131012012310220133103, \\ 20303120203131212032312220333123, \\ 22103300221133012212330222133303, \\ 22303320223133212232332222333323 \}$$

Note that one member has wildcards and the others do not. We conjecture that combining this with an appropriate alternating De Bruijn sequence (as described in [3] will produce an uptorus without well defined diamondicity.

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