Universal Partial Cycles in Higher Dimensions

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Introduction

This project studies the combination of two generalizations of the De Bruijn Cycle: the De Bruijn torus and the universal partial cycle. We show the existence of novel objects – universal partial matrices and tori – and describe some of their properties.

Definition (Universal Partial Matrix)

A *universal partial matrix* – shortened to "upmatrix" – of sub-matrix size $w \times \ell$ over alphabet $\mathcal{A} \cup \{\diamond\}$ is an array where the rows and columns of the array are *not* cyclic and in which every rectangular array of size $w \times \ell$ over alphabet $\mathcal{A}$ appears exactly once, treating $\diamond$ as matching any symbol in $\mathcal{A}$.

Definition (Universal Partial Torus)

A *universal partial torus* – shortened to "uptorus" – of sub-matrix size $w \times \ell$ over alphabet $\mathcal{A} \cup \{\diamond\}$ is an array where both the rows and columns of the array are considered cyclically and in which every rectangular array of size $w \times \ell$ over alphabet $\mathcal{A}$ appears exactly once, treating $\diamond$ as matching any symbol in $\mathcal{A}$. (c.f. [4] and [3])

Definition (Universal Partial Family)

A *universal partial family* for $\mathcal{A}^\ell$ is a set $\mathcal{S}$ of cyclic strings on $\mathcal{A} \cup \{\diamond\}$ such that every element of $\mathcal{S}$ has the same length, and every word in $\mathcal{A}^\ell$ is covered by exactly one element of $\mathcal{S}$, and is only covered once in that element.

Examples of Upmatrices

Given $\mathcal{A} = \{0, 1\}$ and sub-matrix size $w \times \ell = 2 \times 2$, there are universal partial matrices with two $\diamond$s:

$$\begin{matrix} 0 & 0 & 1 & 1 & 0 \\ 0 & \diamond & 1 & \diamond & 0 \\ 1 & 0 & 0 & 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & \diamond & 1 & \diamond \\ 1 & 0 & 0 & 1 & \diamond \end{matrix}$$

Note that the left upmatrix can be made cyclic in one dimension while the right upmatrix cannot.

Minimal Example of an Uptorus

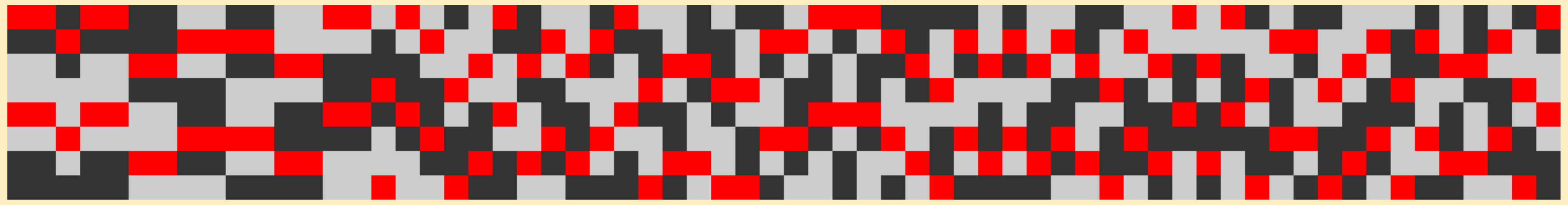
Given $\mathcal{A} = \{0, 1\}$ and sub-matrix size $w \times \ell = 2 \times 2$, there is a universal partial torus with one $\diamond$:

$$\begin{matrix} \diamond & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}$$

This uptorus is minimal in that no uptorus can have fewer $\diamond$s (as it would then be a De Bruijn torus), no nontrivial uptorus can cover a smaller sub-matrix (as it would then be a universal partial cycle), and no nontrivial uptorus can have smaller dimensions (as it would either repeat sub-matrices or omit sub-matrices).

Example Uptorus: An 8×64 Uptorus Covering 4×3 Sub-Matrices over the Binary Alphabet

Black pixels represent 0s, gray pixels 1s, and red pixels $\diamond$s. This construction uses an alternating De Bruijn sequence constructed from the upcycle 001⋄110⋄ (mentioned in [2; 3]) and a De Bruijn cycle generated at [5].



Analogue of the Construction by Kreitzer et al.

Kreitzer et al. [4] constructed De Bruijn tori using alternating De Bruijn sequences and De Bruijn families. A De Bruijn family is like a universal partial family, but it has no wildcards. The alphabet pair that the alternating De Bruijn sequence uses is $(\mathcal{S}, \Pi)$, where $\mathcal{S}$ is the De Bruijn family and Π is the set of rotations on each element of $\mathcal{S}$. The construction involves cyclically shifting elements of $\mathcal{S}$ according to the alternating De Bruijn sequence and then stacking them.

We found that the method works when $\mathcal{S}$ is replaced with a universal partial family. The result includes wildcards and should be an uptorus. An upcycle can be viewed as a universal partial family with a single element.

We have conjectured conditions that should guarantee that the process produces an uptorus and computationally verified a few examples.

Constructing Uptori

Using a method analogous to John C. Cock[1] and Kreitzer et al.'s[4], we can construct an uptorus for alphabet $\mathcal{A}$ and sub-matrix size w rows by ℓ columns, provided that: an upcycle exists for $\mathcal{A}^\ell$ and $w \geq 3$.

1. Construct upcycle u for $\mathcal{A}^\ell$.
2. Construct De Bruijn sequence d for alphabet $\{0 \ldots |u| - 1\}^{w-1}$.
3. Construct a torus by stacking rotated copies of u as rows.

Each row of the uptorus is a rotated copy of upcycle u. The first row is simply u. The j^{th} row is a forward rotation of the $(j-1)^{\text{th}}$ row by a number of characters equal to d_{j-1}.

Universal Partial Families

In [2], the authors identify an upcycle over the alphabet $\{0, 1, 2, 3\}$ for windows of length 4:

001⋄110⋄003⋄112⋄021⋄130⋄023⋄132⋄201⋄310⋄203⋄312⋄221⋄330⋄223⋄332⋄

We conjectured that we could produce a universal partial family by slicing that into equally sized cyclic strings and verified our conjecture by computer for cyclic strings of length 8, 16, and 32 (the slicing works whatever index is chosen for the first cut). For example, the four-member family

001⋄110⋄003⋄112⋄, 021⋄130⋄023⋄132⋄,
201⋄310⋄203⋄312⋄, 221⋄330⋄223⋄332⋄

is a universal partial family. The same slicing construction failed on the binary upcycle for words of length 8:

0000010⋄1111101⋄0010010⋄1101101⋄1110000⋄0001111⋄1110011⋄0101100⋄
1110010⋄0101001⋄1000110⋄0100001⋄1011110⋄0101101⋄0000110⋄1101001⋄

so we conjecture that the slicing construction produces universal partial families if the original upcycle is a product of the alphabet multiplier theorem.

Conclusions and Future Work

We are composing a proof that the construction we describe produces uptori under suitable hypotheses and that (consequently) there exist infinitely many uptori.

We also have many open questions that we hope to answer:

1. Is there an uptorus for the 3×2 window over a binary alphabet? If not, why not?
2. When does the uptorus version of Kreitzer et al.'s method work? For what alphabet and submatrix sizes?
3. Are there infinitely many uptori not constructable by the method similar to Kreitzer et al.'s?
4. Is there a mapping between De Bruijn tori for a particular sub-matrix size and alphabet and uptori for the same sub-matrix size and alphabet?

References

[1] John C. Cock. Toroidal tilings from de bruijn-good cyclic sequences. *Discrete Mathematics*, pages 209–210, 1986.

[2] Dylan Fillmore, Bennet Goeckner, Rachel Kirsch, Kirin Martin, and Daniel McGinnis. The existence and structure of universal partial cycles, 2023. `arXiv:2310.13067`.

[3] Bennet Goeckner, Corbin Groothuis, Cyrus Hettle, Brian Kell, Pamela Kirkpatrick, Rachel Kirsch, and Ryan Solava. Universal partial words over non-binary alphabets. *Theoretical Computer Science*, 713:56–65, 2018. `doi:10.1016/j.tcs.2017.12.022`.

[4] Matthew Kreitzer, Mihai Nica, and Rajesh Pereira. Using alternating de Bruijn sequences to construct de Bruijn tori, 2023. `arXiv:2306.01498`.

[5] Joe Sawada. De Bruijn sequence and universal cycle constructions, 2023. URL: `http://debruijnsequence.org`.