Embeddings of a Semi-Homogeneous Tree in the Hyperbolic Disk

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April 19th, 2024

Introduction

A semi-homogeneous tree is a rooted tree T with root e such that the vertices of odd length have degree d_1 and the vertices of even length have degree d_2, where for $j = 1, 2$, d_j is an integer greater than 2. If $d = d_1 = d_2$, the tree T is homogeneous of degree d. The embeddings of a homogeneous tree have been studied in [3] when d is even, and some of the results in that work have been extended experimentally when d is odd in work in progress.

Notation: T_d denotes a homogeneous tree of degree d and T_{d_1,d_2} denotes a semi-homogeneous tree of degrees d_1, d_2

Definition (Hyperbolic Metric and Geodesics)

The mapping $\rho : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ defined by

$$\rho(z,w) = \frac{1}{2}\ln\left[\frac{1+\left|\frac{z-w}{1-\bar{z}w}\right|}{1-\left|\frac{z-w}{1-\bar{z}w}\right|}\right] \quad \text{for} \quad z, w \in \mathbb{D},$$

is a distance on $\mathbb{D}$ called the hyperbolic or Poincaré metric.
In particular,

$$\rho(z,0) = \frac{1}{2}\ln\left[\frac{1+|z|}{1-|z|}\right] \quad \text{for} \quad z \in \mathbb{D}.$$

The geodesics (i.e. the lines of minimal length) are either circular arcs orthogonal to the unit circle $\partial\mathbb{D}$ or diameters if they contain the origin.

Definition (Embedding)

An embedding of a semi-homogeneous tree T into the disk $\mathbb{D}$ is a injective function $\Phi : T \to \mathbb{D}$ that maps every edge of T onto a geodesic arc having a suitable prescribed hyperbolic length. We assume that $\Phi(e) = 0$.

Definition (Optimal Embedding)

An optimal embedding of T into $\mathbb{D}$ is a one-to-one mapping of the tree into $\mathbb{D}$ such that the set of limit points of the vertices of T is the entire unit circle and the edges of T are geodesic curves of the same hyperbolic length.

Rigidity in the choice of edge lengths

In the homogeneous case, we had only one choice of the Euclidean length r_0 of the vertices of length 1 that would give rise to the optimal embedding. For $0 < r < r_0$, there is no embedding, while for $r_0 < r < 1$, the set of limit points of the vertices covers only a portion of the unit circle.
When considering the semi-homogeneous case, we conjectured that there might be an extra degree of freedom in the choice of edge lengths relative to the two degrees.

Theorem

There is no degree of freedom in the choice of edge length relative to the two degrees.

The location of the vertices is uniquely determined by the coordinate of the vertices of length 1.

Proof in the even-degree case.

Let r be the coordinate on the positive real axis of the vertex of length 1 and let s be the corresponding coordinate of the vertex of length 2. The edges $[0, r]$ and $[r, s]$ have the same hyperbolic length, so

$$\frac{1}{2}\log\left[\frac{1+r}{1-r}\right] = \frac{1}{2}\log\left[\frac{1+\frac{s-r}{1-sr}}{1-\frac{s-r}{1-sr}}\right]$$

which gives

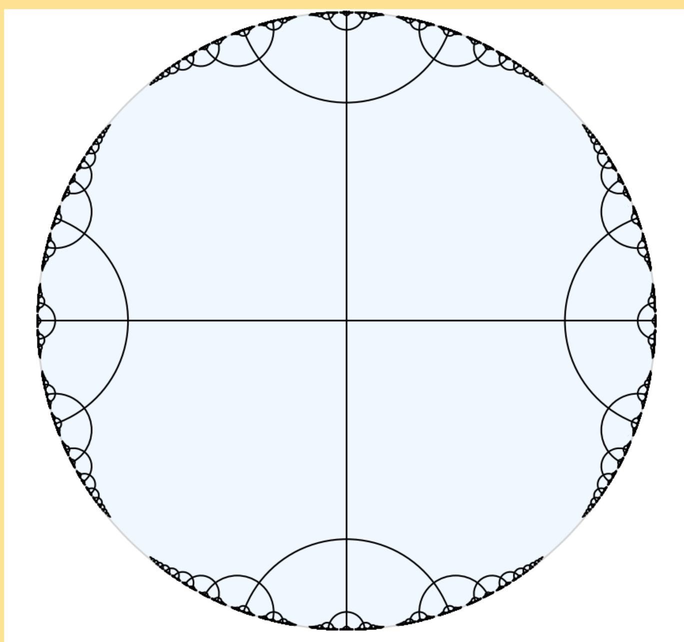
$$s = \frac{2r}{1+r^2}. \quad \text{Therefore,} \quad r = \frac{1-\sqrt{1-s^2}}{s}.$$

To gain some inspiration on how to treat the problem we looked at a "degenerate" semi-homogeneous tree.

The (4,2)-tree

The (4, 2)-tree T can be viewed as a homogeneous tree T_4 of degree 4 all of whose edges have been split into two adjacent geodesic arcs of the same hyperbolic length and the new vertices created from this split have been added to the set of vertices of the tree T. In order to obtain an optimal embedding of the tree T into $\mathbb{D}$, it suffices to apply the optimality results from [3] to the associated homogeneous degree tree T_4. The desired condition is

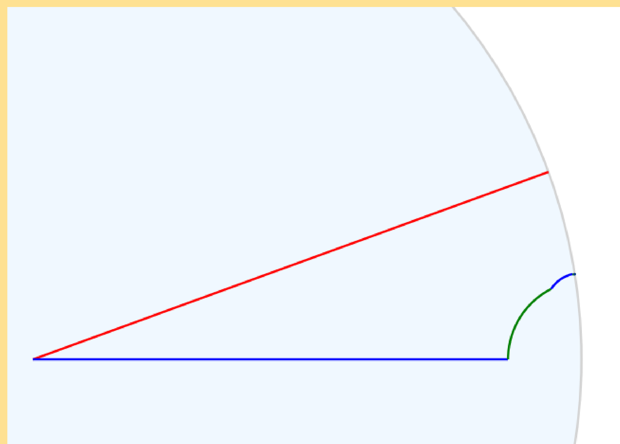
$$s = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$



Strategies for the general case when the degrees are both even. Look at $T_{(6,4)}$: a toy example

Our idea was to compare $T_{(6,4)}$ to the homogeneous tree T_{18}.
[Motivation: $18 = 6 \cdot (4-1)$ is the number of vertices of length 2 of $T_{(6,4)}$.]
The coordinate of the vertex of length 1 of T_{18} that gives an optimal embedding is $s = \cos(\frac{\pi}{18})$. We conjecture that the value of r that gives the optimal embedding for $T_{(6,4)}$ is

$$r = \sec\frac{\pi}{18} - \tan\frac{\pi}{18}.$$



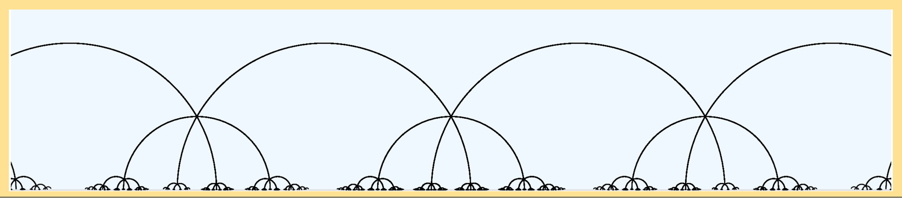
This image shows the the first three outward edges in relation to the straight geodesic of the comparison tree T_{18}. There seems to be a significant gap, but we could not verify this through an algorithm.

Half Plane Representation

The map $V(z) : \mathbb{D} \to \mathbb{H}$

$$\frac{-iz+1}{z-i}$$

maps points in the unit disk to points in the upper half plane. It is a Möbius transformation that combines a circle inversion about a circle centered at i and passing through -1,1 with reflection about the x-axis.



Difficulties encountered

- Poor visualization – From this image it appears that the inner extremes of the left and right subtrees approach each other, but the forced small number of iterations and the smaller Euclidean size of the edges as one moves toward the boundary do not allow to see whether or not gaps will form.
- Group representation to construct the embedding not known.

Conclusions/Future Work

Our first goal for future work is to understand the group structure associated with the semi-homogeneous tree to then derive a group-based approach to the determination of the optimal embedding and an algorithm in the cases studied in this project. We then wish to obtain a visualization of the optimal embedding in the general case and give a formal proof for a publication.

Acknowledgments

We would like to thank Robert Allen, for his talk on Embeddings of trees in the Hyperbolic disk, which led to a break through in the degree 4 case.
Additionally, we would like to thank Dr. Sean Lawton for his suggestions on how to create visualizations in the Poincaré disk.
Finally, thank you to our faculty mentor, Dr. Flavia Colonna and graduate mentor, Madeline Horton.

References

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Degree six non-embedding