Notes on Categories and Finite 2-Groups

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Chapter 1

Background

1.1 Categories

1.1.1 Introduction

Definition 1.1.1 (Category)**.** We define a *category* $\mathscr{C}$ to be a collection of objects and morphisms (or arrows) which relate pairs of objects. Morphisms are composable, and for each object A, there exists an identity morphism id_A which relates A to itself. Moreover, $\mathscr{C}$ satisfies the following axioms.

1. If $f : A \to B$ and $f : A' \to B'$, then $A = A'$ and $B = B'$.
2. If $f : A \to B$ and $g : B \to C$, then $gf : A \to C$.
3. $id_A : A \to A$
4. $f(gh) = (fg)h$
5. $id \circ f = f = f \circ id$

Note: A morphism need not be a function.

Example 1.1.1. Let $G = (V, A)$ be a transitive directed graph with vertices V and paths A. Then G constitutes a category with objects V and morphisms A.

Example 1.1.2. Let V be a set of sets. Define $\mathbf{Ens}_V$ to be the category with objects being sets in V, and morphisms being the collection of all functions between sets in V.

Definition 1.1.2. Let $\mathscr{C}$ be a category. We define $\text{Ob}(\mathscr{C})$ to be the objects of $\mathscr{C}$, and $\text{Hom}(\mathscr{C})$ to be the morphisms of $\mathscr{C}$. Notice that we do not refer to these collections as "sets" because technically the collection of objects and morphisms of a category need not be sets. If both $\text{Ob}(\mathscr{C})$ and $\text{Hom}(\mathscr{C})$ are sets, then we say $\mathscr{C}$ is a *small category.*

Example 1.1.3. Let M be a monoid. Then M is a category in for any category D and any object $* \in D$, we have that $Hom(*, *)$ forms a monoid under composition. That is, the composition of morphisms from $*$ to itself are associative. Alternatively, we can think of any monoid M as a one object category with the set of morphisms from the object to itself representing the elements of M. In this sense, we can think of a category as a kind of generalized monoid.

Example 1.1.4. Let G be a group. Then G is a small category containing a single object where each morphism is invertible. Here the morphisms are thought of as the group elements.

To avoid certain set-theoretic paradoxes, we will suppose that there exists some sufficiently large set U called the universe which will allow us to describe all small categories in terms of sets. We then define any set inside U to be a small set.

Example 1.1.5. Set is the small category whose objects are all small sets, and whose morphisms are all functions between them.

Example 1.1.6. Top is the small category whose objects are all topological spaces, and whose morphisms are all continuous maps between them.

Example 1.1.7. **Cat** is the category whose categories are all small categories, and whose morphisms are all morphisms between them.

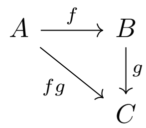
Example 1.1.8. **Grp** is the category of all small groups with morphisms being all group homomorphisms.

Example 1.1.9. **CRng** is the category of all small commutative rings and their homomorphisms.

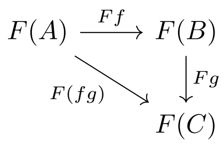
1.1.2 Functors

Definition 1.1.3. Let $\mathscr{C}$,$\mathscr{B}$ be categories. A **functor** T consists of two functions $T : Ob(\mathscr{C}) \to Ob(\mathscr{B})$ and $T : Hom(\mathscr{C}) \to T : Hom(\mathscr{B})$. To elaborate on the second function, given any morphism $f : c \to c'$ of $\mathscr{C}$, we define $Tf : Tc \to Tc'$ such that $T(I_c) = I_{Tc}$ and $T(gf) = Tg \circ Tf$ whenever f, g are composable morphisms.

Example 1.1.10. Suppose that a category $\mathscr{C}$ has $Ob(\mathscr{C}) = \{A, B, C\}$ and $Hom(\mathscr{C}) = \{f, g\}$ related via the following commutative diagram.



Then we can define a functor F which maps $\mathscr{C}$ to the following category.



Example 1.1.11. The map $\mathscr{P} : \textbf{Set} \to \textbf{Set}$ is a functor whose object function takes each $X \in \textbf{Set}$ to the usual power set of X, and whose arrow function takes each $f : X \to Y$ to the function $\mathscr{P}f : \mathscr{P}x \to \mathscr{P}Y$ which maps the function f to the function $\mathscr{P}f$ which takes each $S \subseteq X$ to $f(S) \subseteq Y$.

Example 1.1.12. Let R be a commutative ring. Recall that the general linear group $GL_n(R)$ is the set of all $n \times n$ non-singular matrices with entries from R. Observe that if $\Phi : R \to R'$ is a ring homomorphism, then we naturally produce a group homomorphism $GL_n\Phi : GL_n(R) \to GL_n(R')$. This defines a functor $GL_n : \textbf{CRng} \to \textbf{Grp}$.

Note: Mostly taken from Maclane, *Categories for the Working Mathematician*[5]

1.2 Strict monoidal categories

The first way of intuiting the notion of a (strict) 2-Group is that it is a strict monoidal category in which all objects and morphisms are invertible. To understand this, we first define what a strict monoidal category is.

1.2.1 Monoidal categories

Recall that a monoid is a set M together with a composition law

$$M \times M \to M$$

which is unital and associative. We will attempt to slightly generalize the notion of associativity.

Example 1.2.1. Let V, U be vector spaces over a field K, and let $b : V \times U \to W$ be a bilinear map. We say that b is *universal* if for (come back to).

Nonunital monoidal categories

Definition 1.2.1. Let $\mathscr{C}$ be a category. A **nonunital strict monoidal structure** on $\mathscr{C}$ is a functor

$$\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C} \tag{1.1}$$

such that for any objects $A, B, C \in \mathscr{C}$, $A \otimes (B \otimes C) = (A \otimes B) \otimes C$, and for any morphisms $f, g, h \in \mathscr{C}$, $f \otimes (g \otimes h) = (f \otimes g) \otimes h$.

Note: In (1), the symbol $\times$ is **not** the ordinary cartesian product. Instead, it is a kind of product of categories which can be thought of as two different mappings:

$$(\mathscr{C} \times \mathscr{C})_o = Ob(\mathscr{C}) \times Ob(\mathscr{C})$$
$$(\mathscr{C} \times \mathscr{C})_m = Hom(\mathscr{C}) \times Hom(\mathscr{C})$$

where $\times$ here is just the usual cartesian product.

With this, we simply define a nonunital strict monoidal category to be a category $\mathscr{C}$ together with a nonunital strict monoidal structure $\otimes$ on $\mathscr{C}$, denoted by $(\mathscr{C}, \otimes)$.

Strict unital monoidal categories

Definition 1.2.2. We define a *strict unital monoidal structure* on a category $\mathscr{C}$ to be a non-unital monoidal structure on $\mathscr{C}$ such that there exists an object $1 \in \mathscr{C}$ satisfying:

1. $A \otimes 1 = A = 1 \otimes A$ for all objects $A \in \mathscr{C}$.
2. $f \otimes id_1 = f = id_1 \otimes f$ for all morphisms $f \in \mathscr{C}$.

We define a strict unital monoidal category $(\mathscr{C}, \otimes)$ to be a category $\mathscr{C}$ equipped with a strict unital monoidal structure $\otimes$.

With this new definition we can finally present our first definition of the 2-Group.

Definition 1.2.3. A 2-Group is a strict unital monoidal category in which all objects and morphisms are invertible.

Before providing our first example of a 2-Group, there are a few more terms we need to define.

Definition 1.2.4. A *natural transformation* is a morphism between functors, such that for functors $F, G : \mathscr{C} \to \mathscr{D}$, there exists $\eta : F \to G$ which preserves this structure below for all objects X_1, X_2 in $\mathscr{C}$ and $f : X_1 \to X_2$.

$$\begin{array}{ccc} F(X_1) & \xrightarrow{Ff} & F(X_2) \\ \downarrow{\scriptstyle \eta_{X_1}} & & \downarrow{\scriptstyle \eta_{X_2}} \\ G(X_1) & \xrightarrow{Gf} & G(X_2) \end{array}$$

In other words, there exists $\eta_X : F(X) \to G(X)$ for all $X \in Ob(\mathscr{C})$, which preserves the morphisms Ff to corresponding Gf. [1]

Definition 1.2.5. A natural transformation η is said to be a *natural isomorphism* if for every object $X \in \mathscr{C}$, the morphism η_X is an isomorphism.

Definition 1.2.6. Categories $\mathscr{C}$ and $\mathscr{D}$ are said to have an *equivalence* if there exist functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ where GF is isomorphic to $id_{\mathscr{C}}$ and FG is isomorphic to $id_{\mathscr{D}}$.

1.3 Crossed modules

Towards a second equivalent definition of a 2-Group[2][8], and one that should be easier to approach from a group-theoretic perspective, we will begin our investigation of crossed modules.

Definition 1.3.1. Let H, G be groups. Then consider the following diagram

$$H \underset{\alpha}{\overset{\delta}{\rightleftarrows}} G$$

where δ is a group homomorphism, and α is a group action of G on X where X is the underlying set for the group H. This diagram represents a *crossed module* if given any $g \in G$ and $h \in H$, we have

$$\delta(g \cdot h) = g\delta(h)g^{-1} \tag{1.2}$$

$$\delta(h_1) \cdot h_2 = h_1 h_2 h_1^{-1} \tag{1.3}$$

Recall that every group acts on itself by conjugation. Then, (2) says that δ is *equivarient* with respect to the conjugation action of G on itself, and (3) is an addition restriction known as the *Peiffer identity.*

Example 1.3.1. Let G be a group and let H be normal in G. Then in the diagram above, let δ be the inclusion map, and let α be the conjugation action of G on H. Then this constitutes a crossed module.

Proof. Let $g \in G$ and $h \in H$. Clearly δ is a group homomorphism, and by construction α is a group action. More precisely, we know that α is a well defined map since given any $g \in G$ and $h \in H$, $\alpha(g,h) = ghg^{-1} \in H$ since $H \trianglelefteq G$. So we need only check that conditions (2) and (3) are satisfied. Observe that

$$\delta(g \cdot h) = g \cdot h = ghg^{-1} = g\delta(h)g^{-1}$$

and that

$$\delta(h_1) \cdot h_2 = h_1 \cdot h_2 = h_1 h_2 h_1^{-1}$$

as needed. ☐

1.4 Monoidal categories to 2-categories

Another equivalent definition of a 2-Group can be found by *delooping* the monoidal category into a 2-Category. First, we'll outline some definitions (**Note:** These definitions were paraphrased from Kerodon):

1.4.1 2-categories

Definition 1.4.1. A 2-Category $\mathscr{C}'$ is defined by the following:

1. The 2-Category has a collection of objects, denoted as $\mathrm{Ob}(\mathscr{C}')$.
2. A category $\mathrm{Hom}(X,Y)$ exists for each $X, Y \in \mathrm{Ob}(\mathscr{C}')$. Objects in $\mathrm{Hom}(X,Y)$ are called 1-morphisms, and usually denoted as f, g where $f, g : X \to Y$.
3. Morphisms exist between pairs of objects f, g in $\mathrm{Hom}(X,Y)$, which are called 2-morphisms.
4. A composition functor exists, such that for objects X, Y, Z in $\mathrm{Ob}(\mathscr{C}')$:

   $$\circ : \mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$$

   In addition, this composition functor is strictly associative.
5. There exists an identity 1-morphism id_X in $\mathrm{Hom}(X,X)$ for each object X in $\mathrm{Ob}(\mathscr{C}')$. In addition, id_X is a unit for both left composition and right composition.

1.4.2 Delooping of a (strict) monoidal category

Definition 1.4.2. Suppose we have a strict monoidal category $\mathscr{C}$. *Delooping* a monoidal category is, to define it informally, "promoting" each object in the monoidal category as a 1-morphism in a 2-category, and each morphism in the monoidal category to a 2-morphism in a 2-category. We can deloop a strict unital monoidal category into a strict 2-category. For a more formal definition, we define the delooping of a strict monoidal category $\mathscr{C}$ to 2-Category $\mathscr{BC}$ as having these properties:

1. There is only one object, denoted as $\bullet$, in the 2-Category $\mathscr{BC}$.
2. The category $\mathrm{Hom}(\bullet, \bullet)$ in $\mathscr{BC}$ is equal to $\mathscr{C}$. (i.e. Objects in $\mathscr{C}$ are 1-morphisms in $\mathscr{BC}$, and morphisms in $\mathscr{C}$ are 2-morphisms in $\mathscr{BC}$)
3. The composition functor of $\mathscr{BC}$ is the functor of the strict monoidal category ($\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$).
4. The identity 1-morphism $id_\bullet$ is the strict unital object in $\mathscr{C}$ (where the object is usually denoted as $1 \in \mathscr{C}$).

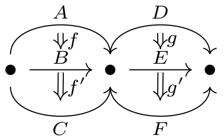
If a 2-Group can be defined as a strict unital monoidal category in which all its objects and morphisms are invertible, a 2-Group can also be defined as a strict 2-Category where:

- All of its 1-morphisms (which was originally the objects of the strict monoidal category) and 2-morphisms (originally the morphisms of the strict monoidal category) are invertible.
- The 2-Category only has one object (due to delooping).

Remark. Note that a property of strict 2-Categories exists called the *interchange law*, such that for 2-morphisms f, f', g, g', and where $\otimes$ represents the functor and $\circ$ represents composition between morphisms:

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g) \tag{1.4}$$

For a visualization[9], consider $\bullet$ as the object in $\mathscr{BC}$, A, B, C, D, E, F as the 1-morphisms in $\mathscr{BC}$ (and the objects of $\mathscr{C}$), and f, f', g, g' as the 2-morphisms in $\mathscr{BC}$ (and the 1-morphisms of $\mathscr{C}$):



With this diagram, a *vertical composition* composes $f : A \to B$ and $f' : B \to C$ with $\circ$ where $f' \circ f : A \to C$ (and likewise with g, g'). A *horizontal composition* composes f, g with the functor $\otimes$. Intuitively, the interchange law says that a horizontal composition composed with a vertical composition is "the same" as a vertical composition composed with a horizontal composition.

For the sake of the proofs below, the interchange law may be used since the strict 2-Group is a strict 2-Category with properties denoted above.

Definition 1.4.3. A *groupoid* is normally defined as a category in which every (1-)morphism is invertible. A *2-groupoid* is a 2-category in which every 1-morphism and 2-morphism is invertible.

(From the categorical definition above, a 2-Group can be defined as a 2-Groupoid with only one object.)

1.5 Strict 2-groups and crossed modules

In this section, we will investigate the two definitions of the 2-Group given so far, and prove that they are equivalent. For conciseness, we will consider the strict unital monoidal category definition to be our strict 2-group. We will show in this section that given any strict 2-Group their exists a crossed module (G, H, t, α), and given any crossed module there exists a strict 2-Group $(\mathscr{C}, \otimes)$.

1.5.1 Strict 2-groups & crossed modules induce each other

Proposition 1. Let $(\mathscr{C}, \otimes)$ be a strict 2-Group. Then we can define a crossed module (G, H, t, α) in the following way:

- Let $G = \mathscr{C}_0$, the collection of objects in $\mathscr{C}$.
- Let H be the set of morphisms emanating from the identity $1 \in (\mathscr{C}, \otimes)$.
- Define the multiplication in both H and G to simply be $\otimes$.
- Let t be defined by $h \mapsto Target(h)$.
- Lastly, define α by $\alpha_g(h) = g \otimes h \otimes g^{-1}$. Recall that we can compute $g \otimes h$ even though g is an object and h is a morphism, since g unambiguously determines the morphism id_g

Proof. We need to show that this defines a crossed module. First, G forms a group under $\otimes$ since $(\mathscr{C}_0, \otimes)$ is a strict unital monoidal structure in which all objects have inverses under $\otimes$. It follows from the functorality of $\otimes$ that H is closed under $\otimes$. That is, tensoring any two unit sourced morphism in $\mathscr{C}_0$ results in another unit sourced morphism in $\mathscr{C}_0$. The remaining group axioms for H follow from the construction of the 2-Group.

It is left to show that t and α are homomorphisms and group actions respectively. Recall how we defined t from our proposition, and how $\otimes$ refers to the operation between objects in G and morphisms in H. We need to show that $t(h_1 h_2) = t(h_1)t(h_2)$ for all $h_1, h_2 \in H$. Recall by the functorality of $\otimes$ that if $h_1 : 1 \mapsto g_1$ and $h_2 : 1 \mapsto g_2$, then $h_1 \otimes h_2 : 1 \otimes 1 \mapsto g_1 \otimes g_2$. Then since the multiplication in H is just given by $\otimes$, we have that $t(h_1 h_2) = t(h_1)t(h_2)$ as needed.

α is a group action because by definition $\alpha_{g_1 \otimes g_2}(h) = g_1 \otimes g_2 \otimes h \otimes g_2^{-1} \otimes g_1^{-1} = \alpha_{g_1}\alpha_{g_2}(h)$ by associativity of $\otimes$, and $\alpha_1(h) = 1 \otimes h \otimes 1^{-1} = h$ by definition of a strict unital monoidal structure, thus satisfying the two properties of a group action.

We now must show that the given crossed module satisfies both equivarience and the Peiffer identity. By our definitions of t and α above, $t(\alpha_g(h)) = t(g \otimes h \otimes g^{-1})$. Noting that g, g^{-1} refers to their identity morphism in this context, it is true that it satifies equivarience, as:

$$\begin{aligned} t(g \otimes h \otimes g^{-1}) &= t(g) \otimes t(h) \otimes t(g^{-1}) \qquad \text{by } t \text{ being a group homomorphism} \\ &= g \otimes t(h) \otimes g^{-1} \end{aligned}$$

As for the Peiffer identity, $\alpha_{t(h_1)}(h_2) = t(h_1) \otimes h_2 \otimes t(h_1)^{-1}$. Note that $t(h_1), t(h_1)^{-1}$ refers to their respective identity morphisms, and their product under $\otimes$ can be composed with the identity morphism for 1. In addition, because the strict 2-Group is a strict 2-Category, the interchange law applies. Therefore:

$$\begin{aligned} t(h_1) \otimes h_2 \otimes t(h_1)^{-1} &= (t(h_1) \otimes h_2 \otimes t(h_1)^{-1}) \circ 1 \\ &= (t(h_1) \otimes h_2 \otimes t(h_1)^{-1}) \circ (h_1 \otimes 1 \otimes h_1^{-1}) \\ &= (t(h_1) \circ h_1) \otimes (h_2 \circ 1) \otimes (t(h_1)^{-1} \circ h_1^{-1}) \qquad \text{by interchange law} \\ &= h_1 \otimes h_2 \otimes h_1^{-1} \end{aligned}$$

Hence α, t satisfies the Peiffer identity. □

Proposition 2. Let (G, H, t, α) be a crossed module. Then we can define a strict 2-Group $(\mathscr{C}, \otimes)$ in the following way:

- Let the objects of $\mathscr{C}$ to be the elements of G.
- Let the morphisms of $\mathscr{C}$ be the semi-direct product of G acting on H, denoted as $H \rtimes_\alpha G$, such that $(h, g) \otimes (h', g') = (h\alpha_g(h'), gg')$ for $h, h' \in H, g, g' \in G$.

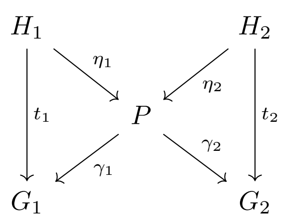
Proof. By G, H being groups, the objects of $\mathscr{C}$ will have inverses and an identity element for $\otimes$. For the identity morphism, 1_H corresponds to the identity morphism id_1 in $\mathscr{C}$. Thus, composition between morphisms in $\mathscr{C}$ is represented by $1_H h = h = h 1_H$ in the crossed module, and inverses of morphisms apply as well since the semi-direct product is itself a group. Thus, a crossed module forms a strict unital monoidal category in which all of its objects and morphisms are invertible. □

1.5.2 Morphisms in the setting of crossed modules

Weak morphisms of crossed modules

Recall that a functor F can be thought of as a morphism between categories, where the functor preserves identity 1-morphisms ($F(id_X) = id_F(X)$ for $X \in \mathscr{C}$) and preserves the composition of 1-morphisms ($F(gf) = F(g) \circ F(f)$). However, the morphism between delooped 2-groups is considered to be a "weak" 2-functor. In the context of crossed modules, the weak 2-functor can be expressed in a *butterfly* diagram. We'll establish some definitions [7][6]:

Definition 1.5.1. A *crossed profunctor* is a commutative diagram between two crossed modules, which we'll denote as $\mathbb{X}_1 = (G_1, H_1, t_1, \alpha)$ and $\mathbb{X}_2 = (G_2, H_2, t_2, \alpha')$, such that the diagram commutes below for group P and for morphisms $\eta_1, \eta_2, \gamma_1, \gamma_2$.



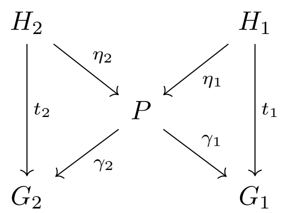
Note that a crossed profunctor has also three characteristics:

- $\gamma_1 \circ \eta_1 = t_1$ and $\gamma_2 \circ \eta_2 = t_2$
- $\gamma_2 \circ \eta_1 = 1$ and $\gamma_1 \circ \eta_2 = 1$
- For $h_1 \in H_1, p \in P, h_2 \in H_2$: $\eta_1(\alpha_{\gamma_1(p)} h_1) = p\eta_1(h_1)p^{-1}, \eta_2(\alpha'_{\gamma_2(p)} h_2) = p\eta_2(h_2)p^{-1}$ (Intuitively, it is like the equivarience property for the crossed module)

We call $H_2 \to P \to G_1$ the *NE-SW sequence* and $H_1 \to P \to G_2$ the *NW-SE sequence.*

Definition 1.5.2. A *butterfly* is a crossed profunctor such that the NE-SW sequence is a *short exact sequence*, where the image of η_2 is equal to the kernel of γ_1, η_2 is a monomorphism, and γ_1 is an epimorphism. A butterfly corresponds to a weak morphism between crossed modules from crossed module $\mathbb{X}_1$ to crossed module $\mathbb{X}_2$.

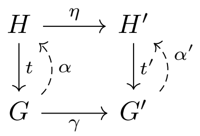
Remark. If the NW-SE sequence is also a short exact sequence, then the diagram is invertible, such that there exists a weak inverse that commutes in the following diagram below[7]:



Remark. Morphisms between butterflies also exists, where we let $f : P \to P'$ be a natural isomorphism in which the structure of the butterfly is maintained. **Food for thought:** Could we use this as a 2-morphism for a crossed module?

(Stricter) Morphisms of Crossed Modules

In other literature [4], they defined a *morphism of crossed modules* $\mathbb{X}$ and $\mathbb{X}'$, which differs from the butterfly diagram above for weak morphisms, by the diagram below for morphisms $\eta : H \to H'$ and $\gamma : G \to G'$.



The morphisms η, γ satisfy these two conditions:

1. $t'\eta = \gamma t$
2. $\eta(\alpha_g(h)) = \alpha'_{\gamma(g)}\eta(h)$ (intuitively, group actions are preserved under the morphism)

Remark. From this definition of a morphism of crossed modules, a category of crossed modules can be defined. It remains to define a 2-morphism to create a 2-category of crossed modules.

Remark. We can define a *homomorphism of crossed modules* by letting $\eta : H \to H'$ and $\gamma : G \to G'$ be group homomorphisms such that it commutes in a diagram and satisfies the two conditions above.

Example 1.5.1. Suppose that you have a homomorphism of crossed modules for $\mathbb{X} = (G, H, t, \alpha)$, $\mathbb{X}' = (G', H', t', \alpha')$, and homomorphisms $\eta : H \to H'$ and $\gamma : G \to G'$. Consider Kerη and Kerγ. It is true that a crossed module can be formed from Kerη and Kerγ since:

- Kerη and Kerγ are normal subgroups to H and G respectively.
- $t_{|\text{Ker}\eta}$: Ker$\eta \to$ Kerγ is a group homomorphism t that is restircted since $t'\eta = \gamma t$. In other words, given $h \in$ Kerη, $t'(\eta_(h)) = t'(e_{H'}) = e_{G'}$. This means that $\gamma(t(h)) = \gamma(g) = e_{G'}$ for $g \in$ Kerγ. Likewise, if $h, h_0 \in$ Kerη, then $t'(\eta(hh_0)) = t'(\eta(h)\eta(h_0)) = t'(e_{H'}) = e_{G'}$, which must mean that $\gamma(t(hh_0)) = \gamma(g) = e_{G'}$ by the condition for the morphism of crossed modules.
- $\alpha_{|\text{Ker}\gamma}$ is also defined since $\alpha'_{\gamma(g)}\eta(h) = \alpha'_{e_{G'}}e_{H'}$, which implies $\eta(\alpha_g(h))$ is defined for Kerγ and Kerη.

Thus, $(\text{Ker}\gamma, \text{Ker}\eta, t_{|\text{Ker}\eta}, \alpha_{|\text{Ker}\gamma})$ form a crossed module.

1.5.3 Crossed submodules

Note: Definitions taken from M. Ladra and A. R.-Grandjéan, "Crossed Modules and Homology" [4].

We now take a moment to recap what has been discussed so far, and where we would like to go with these ideas. We have introduced the notion of 2-groups as strict unital monoidal categories in which all objects and morphisms are invertible. For now, this is a *strict* 2-Group, and so every object and morphism has a strict inverse with respect to the monoidal structure. We then introduced the notion of a crossed module. After some work, we showed that this seemingly unrelated construction allows us to construct a 2-Group, and given any 2-Group we can construct a crossed module. This allows us to study 2-Groups in a more familiar group theoretic setting. Out first original contribution to this discussion was the notion of an Abelian 2-Group, as well as a theorem relating Abelian 2-Groups to crossed modules in which the underlying groups are Abelian and the action is trivial.

Our next goal is to classify 2-Groups and crossed modules. In order to do this, we first need to define a morphism of crossed modules in a way such that crossed modules form a category.

Definition 1.5.3. For a crossed module (G, H, t, α), we can define a *crossed submodule* (G', H', t', α') if G' is a subgroup of G, H' is a subgroup of H, t' is the same as t with respect to H' and α' is the same as α with respect to G', H'.

Definition 1.5.4. A crossed submodule is a *normal crossed submodule* if:

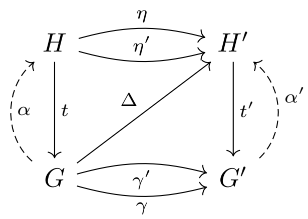
1. G' is also a normal subgroup of G, and H' is also a normal subgroup of H.
2. $\alpha_g(h')$ is in H' for all $g \in G$ and $h' \in H'$
3. $\alpha_{g'}(h)h^{-1}$ is in H' for all $g' \in G'$ and $h \in H$.

Notation wise, it is shown as $(G, H', t', \alpha') \trianglelefteq (G, H, t, \alpha)$.

Definition 1.5.5. Suppose that a crossed module (G, H, t, α) and a crossed submodule (G', H', t', α'). We denote a *quotient crossed module* as $(G, H, t, \alpha)/(G', H', t', \alpha') := (G/G', H/H', \bar{t}, \overline{\alpha})$ where we denote $\bar{t}$ as the boundary operator[1] where $\bar{t}(hH') = t(hH')$ and the group action from G/G' onto H/H' by $\overline{\alpha}_{gG'}(hH') = \alpha_g(h)H'$. [3]

1.5.4 2-morphisms of crossed modules

Suppose you have morphisms between crossed modules $\mathbb{X} = (G, H, t, \alpha)$ and $\mathbb{X}' = (G', H', t', \alpha')$ such that the following diagram commutes below: [8]



A 2-morphism of crossed modules $\Delta : (\eta, \gamma) \to (\eta', \gamma')$ can be thought of as a map that satifies these three conditions:

1. $\Delta(gg^*) = \Delta(g)\alpha'_{\gamma(g)}\Delta(g^*)$ for $g, g^* \in G$
2. $t'(\Delta(g)) = \gamma'(g)\gamma(g)^{-1}$ for $g \in G$
3. $\Delta(t(h)) = \eta'(h)\eta(h)^{-1}$ for $h \in H$

1.6 New territory

In this section we will propose some new definitions and terminology to expand our understanding of the 2-Group. This is in part to pave the way for future attempts to construct group-theoretic analogs for the 2-Group.

1.6.1 Finite abelian 2-groups

Definition 1.6.1. Let $\mathscr{G} = (\mathscr{C}, \otimes)$ be a 2-Group. We say that $\mathscr{G}$ is *Abelian* if $\otimes$ is commutative in $\mathscr{C}$. That is for all $X, Y \in Ob(\mathscr{C})$ and $f, g \in Hom(\mathscr{C})$, $f \otimes g = g \otimes f$ and $X \otimes Y = Y \otimes X$.

Proposition 3. Let $\mathscr{G}$ be a 2-Group with the associated crossed module (G, H, t, α). Then $\mathscr{G}$ is Abelian if and only if H, G are Abelian and α is trivial (such that $\alpha_g(h) = h$ for all $g \in G, h \in H$).

Proof. Suppose that $\mathscr{G}$ is Abelian, which means $\otimes$ is commutative in $\mathscr{C}$ for both objects and morphisms. From our first proposition above, it follows that G constructed by the objects of $\mathscr{C}$ is Abelian, and likewise with H constructed by the morphisms from the unit object in $\mathscr{C}$. It is left to show that α must be trivial. We have defined α to be conjugation by $\otimes$. Then since $\otimes$ is commutative, we have that for all $g \in G$ and $h \in H$, $\alpha_g(h) = g \otimes h \otimes g^{-1} = g \otimes g^{-1} \otimes h = h$.

[1]Terminology in homology for Ladra's paper. There could be a definition out there where we solely focus on the 2-Group setting.

Next, we'll define α as a trivial group action and t as a group homomorphism of targets by Proposition 1. Note that if the group action is trivial, it fulfills the property of equivariance by the following:

$$\begin{aligned} t(\alpha_g(h)) &= t(h) \\ &= t(h) \otimes 1 \quad \text{by the 2-Group being a strict unital monoidal category} \\ &= t(h) \otimes g \otimes g^{-1} \quad \text{as all objects in the 2-Group are invertible} \\ &= g \otimes t(h) \otimes g^{-1} \quad \text{by the 2-Group being Abelian} \end{aligned}$$

If the group action is trivial, it fufills the Peiffer identity:

$$\begin{aligned} \alpha_{t(h_1)}(h_2) &= \alpha_g(h_2) \\ &= h_2 \quad \text{by definition of the trivial action} \\ &= h_2 \otimes 1 \\ &= h_2 \otimes h_1 \otimes h_1^{-1} \\ &= h_1 \otimes h_2 \otimes h_1^{-1} \quad \text{by the 2-Group being Abelian} \end{aligned}$$

To prove the converse, suppose that G, H are Abelian and α is trivial. It follows from the construction of $\mathscr{G}$ that $\otimes$ is commutative for all objects in $\mathscr{G}$. It is left to show that $f \otimes g = g \otimes f$ for all morphisms $f, g \in \mathscr{G}$. Recall from Proposition 2 that the morphisms of $\mathscr{G}$ are defined to be $H \rtimes_\alpha G$, where multiplication is defined to be $(h,g) \otimes (h',g') = (h, \alpha_g(h'), gg')$. Since α is trivial, we simply have $(h,g) \otimes (h',g') = (h, \alpha_g(h'), gg') = (hh', gg')$. Lastly, since H, G are Abelian, we have that $(hh', gg') = (h'h, g'g) = (h', g') \otimes (h, g)$ as needed. □

Example 1.6.1. Suppose that a crossed module is formed where $H = \mathbb{Z}_n$, $G = \mathbb{Z}_2$, t is trivial such that $t(h) = 0_G$ (the identity element for $\mathbb{Z}_2$) and α_g is trivial such that $\alpha_g(h) = h$. By Proposition 2, the elements of G (denoted as $0_G, 1_G$) are the objects of $\mathscr{G}$ and H refers to the unit morphisms of $\mathscr{G}$ (denoted as $0_H, 1_H, ..., (n-1)_H$) whose source is 0_G. Since t is trivial, the target of all unit source morphisms is 0_G. Morphisms are formed by the semi-direct product $H \rtimes_\alpha G$, so $(h,g) \otimes (h',g') = (hh', gg')$. Note that hh', gg' are composed by the addition operation in $\mathbb{Z}_n$ and $\mathbb{Z}_2$ respectively.

1.6.2 Examples of crossed modules to 2-groups

From Proposition 2, we can explore examples of crossed modules and how they can be formed in the strict 2-Group setting.

Example 1.6.2. Suppose that the crossed module is formed where $H = \mathbb{Z}_n, G = \mathbb{Z}_2$, t is trivial, and the group action has the properties where $\alpha_0(h) = h, \alpha_1(h) = -h$. The case where $(h,0) \otimes (h',g') = (hh', g'0) = (hh', g')$ follows as expected similar to trivial actions. The case where $(h,1) \otimes (h',g') = (-hh', 1g')$ is more subtle.

- If $g' = 0$, then $(h,1) \otimes (h',0) = (h(-h'), 1)$
- If $g' = 1$, then $(h,1) \otimes (h',1) = (h(-h'), 0)$

It remains to show that this forms a dihedral group.

Chapter 2

New Results

2.1 Problems

Example 2.1.1. Classify all crossed modules (H, G, t, α) where $|H|, |G| \leq 4$.

Solution: Let G be a group with $|G| \leq 4$. Then G is one of $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We will exclude the case where $|G| = 1$. We recall that any crossed module endowed with the pair (t, α) must satisfy equivarience and the Peiffer identity. Since all groups here are Abelian, we have the following simpler identities:

$$t(g \cdot h) = t(h) \tag{2.1}$$

$$t(h_1) \cdot h_2 = h_2 \tag{2.2}$$

We have the table for $t : H \to G$ as the group homomorphism, where the upper row represents H and the side column G (note that we are *including* the trivial homomorphism, so each entry will be at least 1).

t	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$Z_2 \oplus \mathbb{Z}_2$
$\mathbb{Z}_2$	2	1	2	4
$\mathbb{Z}_3$	1	3	1	1
$\mathbb{Z}_4$	2	1	4	1
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	4	1	4	16

For the first column under $\mathbb{Z}_2$, the entries are they way they are is because homomorphisms must map the identity of H to G, which leaves 1 in $\mathbb{Z}_2$ to be mapped to an element in G. Thus, for the homomorphism $t(0+1)$:

$$t(0 + 1) = t(1) = t(0) + t(1)$$

For the homomorphism $t(1 + 1)$:

$$t(1 + 1) = t(0) = 0 = t(1) + t(1)$$

Developing Intuition: Homomorphisms and Cyclic Groups

We assert that another way to determine a sufficient homomorphism is that a cyclic group of order n from a generator in H could also map to a cyclic group of order n in G. However, the trivial homomorphism maps every element, and thus every cyclic group, to 0 in G, the cyclic group of order 1. How can we generalize such a characteristic for cyclic subgroups in G under the homomorphism t?

Proposition 4. Suppose H, G are finite groups. Let $t : H \to G$ be a homomorphism, and let $h \in H, g \in G$. Define the cyclic groups $\langle t(h) \rangle = \langle g \rangle$. Then the order of $\langle g \rangle$ must be a divisor of the order of $\langle h \rangle$.

Proof. Recall that a cyclic group $\langle g \rangle$ is defined by $\{g^m : g \in G, m \in \mathbb{Z}\}$. In addition, because G is finite, its cyclic group is finite, so let M be the order of $\langle g \rangle$. Let $m \in \{0, 1, ..., M - 1\}$ so that $\langle g \rangle = \{e, g, ..., g^{M-1}\}$. Since $g = t(h)$, then $g^m = t(h)^m = t(h^m)$. Since there are M distinct vectors in $\langle g \rangle$, this means that $\langle h \rangle$ has at least M elements.

It is left to show that the order of $\langle h \rangle$ is a multiple of M. Note that from above that if $t(h) = g$, then $t(h^{M-1}) = g^{M-1}$. Suppose that $h^M \neq e \in H$. However, $t(h^M) = t(h^{M-1})t(h) = g^{M-1}g = e$. Likewise, for any h^{kM} for $k \in \mathbb{N}$, it is true that $t(h^{kM}) = t(h^M)^k = e^k = e$. Suppose there exists h^{kM+r} for $r \in \mathbb{N}$, and $r \leq M - 1$, such that $h^{kM+r}h = e$. Then it is true that $e = t(h^{kM+r}h) = t(h^M)^k t(h^r)t(h) = eg^r g$, which implies $r = M - 1$, thus $\langle h \rangle$ must have an order that is a multiple of M. □

Remark. From the proposition above, it becomes easier to classify homomorphisms between groups, especially groups of different orders. For example, we assert that $\mathbb{Z}_2 \to Z_3$ has no non-trivial homomorphism, as $\langle 1 \rangle$ in $\mathbb{Z}_2$ is a cyclic group of order 2, but $\langle 1 \rangle, \langle 2 \rangle$ in $\mathbb{Z}_3$ are cyclic groups of order 3. If one were to assert that $t(1) = 1 \in \mathbb{Z}_3$, then t would not be a group homomorphism because:

$$t(1+1) = t(0) = 0 \neq t(1) + t(1) = 1 + 1 = 2$$

Likewise, if one were to assert that $t(1) = 2 \in \mathbb{Z}_3$, then:

$$t(1+1) = t(0) = 0 \neq t(1) + t(1) = 2 + 2 = 1$$

In contrast, $t : \mathbb{Z}_2 \to \mathbb{Z}_4$ has one non-trivial homomorphism because we can let $t(1) = 2 \in \mathbb{Z}_4$, and $t(1+1) = t(0) = 0 = 2 + 2 = t(1+1)$, since $\langle 2 \rangle \in \mathbb{Z}_4$ is of order 2.

Meanwhile, $\mathbb{Z}_3 \to \mathbb{Z}_2$ has no non-trivial homomorphisms because if $t(2) = 1$, then $t(1) = t(2+2) = t(2) + t(2) = 1 + 1 = 0$, but this implies $0 + 0 = t(1) + t(1) = t(1+1) = t(2)$. Likewise, if $t(1) = 1$, then $t(2) = t(1+1) = t(1) + t(1) = 1 + 1 = 0$, but $0 + 0 = t(2) + t(2) = t(2+2) = t(1)$.

What about $\mathbb{Z}_3 \to \mathbb{Z}_3$? If $t(1) = 1, t(2) = 2$, then it is just the identity homomorphism. If $t(1) = 2$, $t(2) = 1$, then:

- $t(1+2) = t(0) = 0 = 2 + 1 = t(1) + t(2)$
- $t(1+1) = t(2) = 1 = 2 + 2 = t(1) + t(1)$
- $t(2+2) = t(1) = 2 = 1 + 1 = t(2) + t(2)$

We can do this because the order of $\langle 1 \rangle$ is equal to the order of $\langle 2 \rangle$ for $1, 2 \in \mathbb{Z}_3$.

Continuing forward, for homomorphisms from $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, testing more of our intuition, would mapping all elements of order 2 to $1 \in \mathbb{Z}_2$ be a homomorphism? Let $(1,0) \to 1, (0,1) \to 1, (1,1) \to 1$. However, we have:

$$0 = 1 + 1 = t((1,0)) + t((0,1)) \neq t((1,0) + (0,1)) = t((1,1)) = 1$$

What about if we let $(1,1) \to 0$ instead? We would have:

$$1 = 1 + 0 = t(0,1) + t(1,1) = t((0,1) + (1,1)) = t((1,0)) = 1$$

So the homomorphism that sends $(0,0) \to 0, (1,0) \to 1, (0,1) \to 1$, and $(1,1) \to 0$ seems to fit!

Developing Intuition: Cyclic Groups Continuation and Generators

It's interesting how haphazardly mapping everything to an element of the same order doesn't always give a group homomorphism! Why is this so?

For reference, consider the Cayley tables for $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and Z_2 respectively, for which the groups have the same order.

$\mathbb{Z}_2 \otimes \mathbb{Z}_2$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

$\mathbb{Z}_2$	0	1
0	0	1
1	1	0

Cyclic groups can be generated by one element, but how about groups generated by two elements? For example, consider the group generated by both $\{(0,1), (1,0)\}$ for $(0,1), (1,0) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$. A group would be of order 4. However, you would need two generators in order for such a group to be equal to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, whereas $\mathbb{Z}_2$ only needs one element (which is 1).

A similar argument can be made for $\mathbb{Z}_4$, where you have one generator (1) where $\langle 1 \rangle = \mathbb{Z}_4$, as shown in the Cayley Table below:

$\mathbb{Z}_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Notice that $\langle 1 \rangle, \langle 3 \rangle$ are of order 4. However, $\langle 2 \rangle$ is of order 2.

For $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, we would only need to consider the maps of the generators in order to get a group homomorphism.

What about $(1,0) \to 1, (0,1) \to 0, (1,1) \to 1$? Or $(0,1) \to 0, (0,1) \to 1, (1,1) \to 1$? They would also be homomorphisms as well! (Refer to Cayley table for $Z_2 \oplus Z_2$ to see why).

2.1.1 Group actions

We have now determined the number of possible group homomorphisms between groups with order less than 5. We now turn our attention to group actions. We want to find the total number of possible group actions between groups of order less than 5. To do this, we recall that a group G acts on a group $\overline{G}$ by an action α, where $\alpha : G \to Aut(\overline{G})$. So it suffices to determine the number of group homomorphisms from G to $Aut(\overline{G})$.

α	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$Z_2 \oplus \mathbb{Z}_2$
$Aut(\mathbb{Z}_2)$	1	1	1	1
$Aut(\mathbb{Z}_3)$	2	1	2	4
$Aut(\mathbb{Z}_4)$	2	1	2	4
$Aut(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$	4	3	4	7

To simplify this table slightly, we will use the following facts from group theory:

$$
\begin{aligned}
Aut(\mathbb{Z}_2) &\cong \mathbb{Z}_1 \\
Aut(\mathbb{Z}_3) &\cong \mathbb{Z}_2 \\
Aut(\mathbb{Z}_4) &\cong \mathbb{Z}_2 \\
Aut(\mathbb{Z}_2 \oplus \mathbb{Z}_2) &\cong S_3
\end{aligned}
$$

We have now determined all group homomorphisms and group actions between groups of order less than five. It remains to determine which pairs (t, α) are compatible with the equivarience and Peiffer identity conditions. Observe that since all groups of order 4 or less are Abelian, it suffices to show that (t, α) satisfy conditions (1) and (2). It can be shown that the trivial homomorphism is compatible with any group action, and the trivial group action is compatible with any homomorphism. The only remaining cases are of the form (t, α) where both t and α are non-trivial.

:)	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$Z_2 \oplus \mathbb{Z}_2$
$\mathbb{Z}_2$	2	2	2	4
$\mathbb{Z}_3$	1	3	2	4
$\mathbb{Z}_4$	2	1		4
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	4			

2.1.2 Cyclic crossed modules vs. abelian crossed modules

Lemma 1. Suppose that for a given crossed module $\mathbb{X} = (G, H, t, \alpha)$, there exists a subgroup K of H. Then $(\text{Im } t|_K, K, t|_K, \alpha|_K)$ forms a crossed submodule of $\mathbb{X}$.

Proof. Let K be a subgroup of H and denote the image of t restricted by K as $\text{Im } t|_K$. Since K is a subgroup, it is true that by definition of t as a homomorphism, $\text{Im } t|_K$ also forms its own subgroup in G. α restricted on the image of K under t, denoted as $\alpha|_K$, respects the Peiffer identity since for $k, k' \in K$, $\alpha_t(k)k' = kk'k^{-1} \in K$. It also respects equivarience, since for $t(h*) \in \text{Im } t|_K$, $t(\alpha_{t(h*)}h) = t(h^* h {h^*}^{-1}) = t(h^*)t(h)t(h^*)^{-1}$. ☐

Proposition 5. Suppose $H = \langle h \rangle$ and $G = \text{Im } t$ in a crossed module $\mathbb{X} = (G, H, t, \alpha)$. Then the crossed module must be abelian.

Proof. Suppose that h is of order M, such that $\langle h \rangle = \{e, h, h^2, \ldots, h^{M-1}\}$. Then it is true that $G = \langle t(h) \rangle$ (see Proposition 1). It is also true that cyclic groups are abelian, so H, G are abelian. By the Peiffer identity, it is true that $\alpha_{t(h)}h' = hh'h^{-1}$. Since H is abelian, $hh'h^{-1} = hh^{-1}h' = h'$. This means that $\alpha_{t(h)}h'$ is trivial. Since all elements of G is the image of t, α is trivial for all elements in G. □

Remark. From the above, if t is not surjective such that $G \neq \text{Im } t$, then α need not be trivial. A counterexample of this is the crossed module $(\mathbb{Z}_2 \oplus Z_2, \mathbb{Z}_4, t, \alpha)$, where t is not surjective and there exists a non-trivial action α.

Proposition 6. Suppose $H = \langle h \rangle, G = \langle g \rangle$ in a crossed module $\mathbb{X} = (G, H, t, \alpha)$. If t is injective, then the crossed module must be abelian.

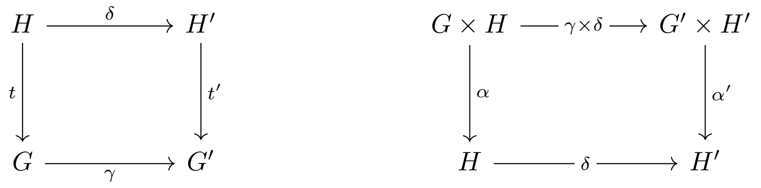
Proof. Since H and G are cyclic groups, H and G are abelian. From the Peiffer identity, for $h, h' \in H$, $\alpha_t(h)h' = h'$ for the image of H under homomorphism t (refer to Proposition 2). From the property of equivarience, $t(\alpha_g h) = gt(h)g^{-1} = gg^{-1}t(h) = t(h)$. Since t is injective, $t(\alpha_g h) = t(h)$ implies that $\alpha_g h = h$. □

Remark. From the proposition above, if t is not injective, then α need not be trivial. A counterexample is the crossed module $(\mathbb{Z}_2, \mathbb{Z}_4, t, \alpha)$, where t is characterized as $\langle 1 \rangle_{\mathbb{Z}_4} \mapsto \langle 1 \rangle_{\mathbb{Z}_2}$. There exists a non-trivial action $\alpha : \mathbb{Z}_2 \to \text{Aut}\mathbb{Z}_4 \cong \mathbb{Z}_2$ where $\alpha_1 1 = 3$ and $\alpha_1 3 = 1$.

2.2 Morphisms of crossed modules

We would like to classify 2-Groups. To do this, we must first define a morphism of crossed modules.

Definition 2.2.1. Let $\mathbb{X} = (H, G, t, \alpha)$ and $\mathbb{X}' = (H', G', t', \alpha')$ be crossed modules. We define a morphism of crossed modules $f : \mathbb{X} \to \mathbb{X}'$ as a pair of group homomorphisms $\gamma : G \to G'$ and $\delta : H \to H'$ such that the following diagrams commute.



In symbols, this means

$$\gamma(t(h)) = t'(\delta(h))$$

and

$$\delta(\alpha_g(h)) = \alpha'_{\gamma(g)}(h)$$

.Moreover, we have the following notions of equivalence.

Definition 2.2.2. Let $(\gamma, \delta) : \mathbb{X} \to \mathbb{X}'$ be a morphism of crossed modules. We say that $f = (\gamma, \delta)$ is an isomorphism if there exists a morphism $f' = (\gamma', \delta') : \mathbb{X}' \to \mathbb{X}$ such that $f' \circ f = id : \mathbb{X} \to \mathbb{X}'$.

Proposition 7. Suppose G is isomorphic to G' and H is isomorphic to H'. Let $\mathbb{X} = (H, G, t, \alpha)$ denote a crossed module. Then there exists a crossed module $\mathbb{X}' = (H', G', t', \alpha')$ such that there exists an isomorphism of crossed modules $\mathbb{X}$ and $\mathbb{X}'$.

Proof. Let $\gamma : G \to G'$, $\delta : H \to H'$ denote isomorphisms, which implies there exists inverses $\gamma^{-1} : G' \to G$ and $\delta^{-1} : H' \to H$ respectively. First, we can define $t' : H' \to G'$ as $t' = \gamma t \delta^{-1}$.
From group theory, note that since G, H, G', H' are groups, $G \times H$ and $G' \times H'$ are also groups, and since γ, η are isomorphisms, then $\gamma \times \eta : G \times H \to G' \times H'$ is an isomorphism, and there exists an inverse morphism $\gamma^{-1} \times \eta^{-1} : G' \times H' \to G \times H$. Thus, we define $\alpha' : G' \times H' \to H'$ as $\alpha' = \delta\alpha(\gamma^{-1} \times \eta^{-1})$. Thus, we can construct a crossed module $\mathbb{X}' = (H', G', t', \alpha')$ such that there exists a morphism of crossed modules $f : \mathbb{X} \to \mathbb{X}'$.
A very similar argument shows that there exists $f' : \mathbb{X}' \to \mathbb{X}$ from the above, defining $t'' = \gamma^{-1}t'\delta$ and $\alpha'' = \delta^{-1}\alpha'(\gamma \times \delta)$. We can say that $f' \circ f = id : \mathbb{X} \to \mathbb{X}$ is a consistent fact because $t'' = \gamma^{-1}t'\delta = \gamma^{-1}(\gamma t \delta^{-1})\delta = (id_G)t(id_H) = t$ and $\alpha'' = \delta^{-1}\alpha'(\gamma \times \delta) = \delta^{-1}(\delta\alpha(\gamma^{-1} \times \eta^{-1}))(\gamma \times \delta) = (id_H)\alpha(id_{G \times H}) = \alpha$. □

Corollary 1. Suppose G is isomorphic to G' and H is isomorphic to H. Then crossed modules $\mathbb{X} = (H, G, t, \alpha)$ and $\mathbb{X}' = (H', G', t', \alpha')$ can be constructed such that there exists a isomorphism of crossed modules $\mathbb{X}$ and $\mathbb{X}'$.

Proof. Let $\gamma : G \to G'$, $\delta : H \to H'$ be group isomorphisms. We can construct the crossed module $\mathbb{X} = (G, H, t, \alpha)$ where t is the trivial homomorphism where $t(g) = e_H$ for $g \in G$ and α is the trivial action for $\alpha_g(h) = h$. We can construct the crossed module $\mathbb{X}' = (H', G', t', \alpha')$ by Proposition 4 above where there exists an isomorphism between $\mathbb{X}$ and $\mathbb{X}'$. □

Remark. Suppose that $\mathbb{X} = (H, G, t, \alpha)$ and $\mathbb{X}' = (H', G', t, \alpha)$ are already defined, and that G is isomorphic to G' and H is isomorphic to H'. Are these conditions sufficient to say that an isomorphism (f) of crossed modules exists (such that we show the diagrams in Definition 2.1 commute for f and f^{-1})?

No: A general counterexample would be the given crossed modules $\mathbb{X} = (H, G, t, \alpha)$ and $\mathbb{X}' = (H', G', t', \alpha)$ where t, α are non-trivial but t', α' are trivial. Let $\gamma : G \to G'$, $\delta : H \to H'$ be isomorphisms. Let h be an element of H where $h \notin \text{Ker } t$. There exists $h' \in H'$ such that $\delta(h) = h'$ since δ is an isomorphism. It is clear that the left diagram in Defintion 2.1 does not commute since $\gamma t(h) = g' \neq e_{G'}$, but $t'\delta(h) = e_{G'}$ since t' is trivial, thus $\gamma t \neq t'\delta$.

Corollary 2. Suppose that $\mathbb{X} = (H, G, t, \alpha)$ and $\mathbb{X}' = (H', G', t', \alpha')$ are crossed modules such that the diagrams in Definition 2.1 commute and $\delta : H \to H'$ and $\gamma : G \to G'$ are isomorphisms. Let $f : \mathbb{X} \to \mathbb{X}'$ be a morphism of crossed modules. Then there exists $f^{-1} : \mathbb{X}' \to \mathbb{X}$ such that $f^{-1} \circ f = id : \mathbb{X} \to \mathbb{X}$.

Proof. We define f^{-1} as the pair of homomorphisms $(\gamma^{-1}, \delta^{-1})$. From there, we can construct crossed module $\mathbb{X}'' = (H, G, t'', \alpha'')$, noting that $t'' : H \to G$ is defined as $t'' = \gamma^{-1}t'\delta$. However, since $f : \mathbb{X} \to \mathbb{X}'$ exists, this means that $t' = \gamma t \delta^{-1}$, thus by substitution, $t'' = \gamma^{-1}(\gamma t \delta^{-1})\delta = t$. Similar logic goes to show that $\alpha'' = \alpha$. Thus, $f^{-1} \circ f = id : \mathbb{X} \to \mathbb{X}$. □

Proposition 8. Every finite Abelian crossed module can be written as the direct sum of cyclic crossed modules of prime order power.

More precisely, we claim that

$$\begin{array}{ccc} H & & \mathbb{Z}_{p_1}^{\alpha_1} \oplus \cdots \oplus \mathbb{Z}_{p_n}^{\alpha_n} \\ t \big\downarrow \big\uparrow \alpha & \approx & (t_1,\dots,t_n) \big\downarrow \big\uparrow (\alpha_1,\dots,\alpha_n) \\ G & & \mathbb{Z}_{q_1}^{\beta_1} \oplus \cdots \oplus \mathbb{Z}_{q_k}^{\beta_k} \end{array}$$

is an equivalence of crossed modules.
Note: If Proposition 4 is true, this will follow immediately as an isomorphism of crossed modules.

2.2.1 Crossed submodules: Kernel and image crossed submodules

Definition 2.2.3. Suppose $\mathbb{X} = (H, G, t, \alpha)$ is a crossed module. Recall that we define a **crossed submodule** $\mathbb{X}' = (H', G', t', \alpha')$ by the following conditions:

- H' is a subgroup of H.
- G' is a subgroup of G.
- For $h' \in H'$, $t'(h') = t(h') \in G'$.
- For $h' \in H'$, $g' \in G$, $\alpha'_{g'}(h') = \alpha_{g'}(h') \in H'$

Lemma 2. A crossed submodule is a crossed module.

Proof. From group theory, subgroups H', G' are groups. Since $t' = t|_{H'}$, it is true that t' is a homomorphism. Since $\alpha' = \alpha$ restricted by elements in G', H', it is true that α' would follow equivarance and the Peiffer identity, since for $g' \in G, h', h'_* \in H$:

- $t'(\alpha'_{g'}(h')) = g't'(h')g'^{-1} \in G'$
- $\alpha'_{t'(h')}(h'_*) = h'h'_*h'^{-1} \in H'$

□

Example 2.2.1. Let $f : \mathbb{X} \to \mathbb{X}'$ be a morphism of crossed modules $\mathbb{X} = (H, G, t, \alpha)$ to $\mathbb{X}' = (H', G', t', \alpha')$, with $f = (\gamma, \delta)$ denoting $\gamma : G \to G'$, $\delta : H \to H'$. Let $\text{Ker}f = (\text{Ker}\gamma, \text{Ker}\delta)$. Then there exists a crossed submodule $\text{Ker}\mathbb{X} = (\text{Ker}\delta, \text{Ker}\gamma, t, \alpha)$.

Proof. The kernel of a group is a subgroup, so $\text{Ker}\delta, \text{Ker}\gamma$ are subgroups of H and G respectively. It is true that $t'\delta(h) = e_{G'}$ for $h \in \text{Ker}\delta$, so this implies that $\gamma t(h) = e_G$. Thus, $t(h) \in \text{Ker}\gamma$. For α, if $h \in \text{Ker}\delta, g \in \text{Ker}\gamma$, then for $(g, h) \in G \times H$, $(\gamma \times \delta)(g, h) = (e_{G'}, e_{H'})$, and $\alpha'_{e_{G'}}(e_{H'}) = e_H$. This implies that $\delta(\alpha_g(h)) = e_H$ (see second diagram in definition of morphism of crossed modules), so $\alpha_g(h) \in \text{Ker}\delta$. □

Example 2.2.2. Let $f : \mathbb{X} \to \mathbb{X}'$ be a morphism of crossed modules $\mathbb{X} = (H, G, t, \alpha)$ to $\mathbb{X}' = (H', G', t', \alpha')$, with $f = (\gamma, \delta)$ denoting $\gamma : G \to G'$, $\delta : H \to H'$. Let $\text{Im}f = (\text{Im}\gamma, \text{Im}\delta)$. Then there exists a crossed submodule $\text{Im}\mathbb{X}' = (\text{Im}\delta, \text{Im}\gamma, t', \alpha')$.

Proof. The range of a group homomorphism is a subgroup of the codomain, so $\text{Im}\delta, \text{Im}\gamma$ are subgroups of H', G' respectively. We know that $\gamma t(h)$ commutes to $\text{Im}\gamma$, which implies that $t'\delta(h)$ must also commute to $\text{Im}\gamma$, so $t'(\delta(h)) \in \text{Im}\gamma$. Same logic applies to show that $\alpha_{\gamma(g)}(\delta(h)) \in \text{Im}\delta$, as $\delta(\alpha_g(h))$ commutes to $\text{Im}\delta$ (refer to second diagram in the definition of morphisms of crossed modules). □

Definition 2.2.4. A **normal crossed submodule** $\mathbb{X}' = (H', G', t', \alpha')$ of a crossed module $\mathbb{X}$ is a crossed submodule that satisfies these three conditions:

- G' is a normal subgroup of G.
- For all $g \in G$ and $h \in H'$, $\alpha_g(h') \in H'$.
- For all $g' \in G'$, $h \in H$, $\alpha_{g'}(h)h^{-1} \in H'$.

Remark. Does the following criteria imply that H' is a normal subgroup of H? Yes, by the Peiffer identity, since $\alpha_{t(h)}(h') = hh'h^{-1} \in H'$ for $h' \in H'$, $h \in H$.

Lemma 3. Suppose $f : \mathbb{X} \to \mathbb{X}'$ is a morphism of crossed modules with $f = (\gamma, \delta)$. A "kernel crossed submodule" denoted as $\text{Ker}f = (\text{Ker}\delta, \text{Ker}\gamma, t, \alpha)$ is a normal crossed submodule of $\mathbb{X}$.

Proof. In group theory, it is known that $\text{Ker}\delta$ is a normal subgroup of H, and $\text{Ker}\gamma$ is a normal subgroup of G. Note that for $g \in G, h \in \text{Ker}\delta$, it is true that $\delta(\alpha_g(h)) = \alpha'_{\gamma(g)}\delta(h) = \alpha'_{\gamma(g)}e_{H'} = e_{H'}$ by the commutative diagram (refer to definition of morphism between crossed modules). Thus, it is true that $\alpha_g(h) \in \text{Ker}\delta$. Now consider $\alpha'_{\gamma(g')}(\delta(h))$ for $g' \in G', h \in H$. Note that $\alpha'_{\gamma(g')}(\delta(h)) = \alpha'_{e_{G'}}(\delta(h)) = \delta(h)$, which implies that $\delta(\alpha_{g'}(h)) = \delta(h)$ by morphism of crossed modules. From there:

$$\begin{aligned}
\delta(\alpha_{g'}(h)) &= \delta(h) \\
\delta(\alpha_{g'}(h))\delta(h^{-1}) &= \delta(h)\delta(h^{-1}) \\
\delta(\alpha_{g'}(h)h^{-1}) &= \delta(hh^{-1}) \\
\delta(\alpha_{g'}(h)h^{-1}) &= \delta(e_H) \\
\delta(\alpha_{g'}(h)h^{-1}) &= e_{H'}
\end{aligned}$$

Which implies $\alpha_{g'}(h)h^{-1} \in \text{Ker}\delta$. Thus, the kernel crossed module is a normal crossed module. □

Definition 2.2.5. Let $\mathbb{X}' = (N_H, N_G, t', \alpha')$ be a normal crossed submodule of $\mathbb{X} = (H, G, t, \alpha)$. A **quotient crossed module** is a crossed module denoted by $\mathbb{X}/\mathbb{X}' = (H/N_H, G/N_G, t^*, \alpha^*)$, where for $h \in H$ and $g \in G$, $t^*(hN_H) = t(h)N_G$, and $\alpha^*_{gN_G}(hN_H) = \alpha_g(h)N_H$.

Proposition 9. A quotient crossed module is a crossed module that is well-defined.

Proof. Since N_H is a normal subgroup of H and N_G is a normal subgroup of G, H/N_H and G/N_G are groups. We must first show that t^* defined in this way is a well defined group homomorphism. In other words, we want to show that if $h_1N_H = h_2N_H$, then $t^*(h_1N_H) = t^*(h_2N_H)$.

Supposing that $h_1N_H = h_2N_H$, there exists $n \in N_H$ where $h_1 = h_2n$ (since $h_1 \in h_1N_H = h_2N_H$). Then we can show that $t^*(h_1N_H) = t(h_1)N_G = t(h_2n)N_G = t(h_2)t(n)N_G = t(h_2)N_G$ (since $t(n) \in N_G$), showing it is well defined.

We also want to show that α^* is well defined, where if $g_1N_G = g_2N_G$, and $h_1N_H = h_2N_H$, then $\alpha^*_{g_1N_G}(h_1N_H) = \alpha^*_{g_2N_G}(h_2N_H)$. Similarly, since $g_1N_G = g_2N_G$, there exists $n \in N_G$ where $g_1 = g_2n$, and likewise there exists $n' \in N_H$ where $h_1 = h_2n'$. Thus:

$$\begin{aligned}
\alpha^*_{g_1N_G}(h_1N_H) &= \alpha_{g_1}(h_1)N_H \\
&= \alpha_{g_2n}(h_2n')N_H \\
&= \alpha_{g_2n}(h_2)\alpha_{g_2n}(n')N_H \\
&= \alpha_{g_2n}(h_2)N_H \qquad (\text{since } \alpha_{g_2n}(n') \in N_H) \\
&= \alpha_{g_2}(\alpha_n(h_2))N_H \\
&= \alpha_{g_2}(\alpha_n(h_2)h_2^{-1}h_2)N_H \\
&= \alpha_{g_2}(\alpha_n(h_2)h_2^{-1})\alpha_{g_2}(h_2)N_H \\
&= \alpha_{g_2}(\alpha_n(h_2)h_2^{-1})N_H\alpha_{g_2}(h_2) \\
&= N_H\alpha_{g_2}(h_2) \qquad (\text{since } \alpha_{g_2}(\alpha_n(h_2)h_2^{-1})) \in H' \\
&= \alpha_{g_2}(h_2)N_H
\end{aligned}$$

Remark. Since N_H is a normal subgroup, $hN_Hh^{-1} = N_H$ for all $h \in H$. Thus, it is true that $hN_H = N_Hh$. Since $\alpha_{g_2}(h_2) \in H$, it is true that $\alpha_{g_2}(h_2)N_H = N_H\alpha_{g_2}(h_2)$

Now that we shown that t^*, α^* is well-defined, we want to show that t^* is a homomorphism and α^* is an action that fulfills equivariance and the Peiffer identity.

For t^*, we need to show that $t^*(h_1N_Hh_2N_H) = t^*(h_1N_H)t^*(h_2N_H)$. For the left-hand side, $t^*(h_1N_Hh_2N_H) = t^*(h_1h_2N_H) = t(h_1h_2)N_G$. For the right-hand side, $t^*(h_1N_H)t^*(h_2N_H) = t(h_1)N_Gt(h_2)N_G = t(h_1)t(h_2)N_G = t(h_1h_2)N_G$, thus showing t^* is a homomorphism.

For α^*, we want to show that $\alpha^*_{g_1N_G}(\alpha^*_{g_2N_G}(hN_H)) = \alpha^*_{g_1N_Gg_2N_G}(hN_H)$ and $\alpha^*_{eN_G}(hN_H) = hN_H$. Note that by definition, $\alpha^*_{g_1N_G}(\alpha^*_{g_2N_G}(hN_H)) = \alpha^*_{g_1N_G}(\alpha_{g_2}(h)N_H) = \alpha_{g_1}(\alpha_{g_2}(h))N_H = \alpha_{g_1g_2}(h)N_H$ for the left-hand side. For the right-hand side, $\alpha^*_{g_1N_Gg_2N_G}(hN_H) = \alpha^*_{g_1g_2N_G}(hN_H) = \alpha_{g_1g_2}(h)N_H$. Likewise, for $\alpha^*_{eN_G}(hN_H)$, we can clearly see that it is equal to $\alpha_e(h)N_H = hN_H$ by definition. Hence, α is a group action.

Then, for α^*, we want to show that it fulfills the equivarience property, such that $t^*(\alpha^*_{gN_G}(hN_H)) = gN_Gt^*(hN_H)g^{-1}N_G$. Looking at the left-hand side, $t^*(\alpha^*_{gN_G}(hN_H)) = t^*(\alpha_g(h)N_H) = t(\alpha_g(h))N_G = gt(h)g^{-1}N_G = gN_Gt(h)N_Gg^{-1}N_G = gN_Gt^*(hN_H)g^{-1}N_G$.

Finally, we want to show α^* fulfills the Peiffer identity, where $\alpha^*_{t^*(h_1N_H)}(h_2N_H) = h_1N_Hh_2N_Hh_1^{-1}N_H$. We can see that $\alpha^*_{t^*(h_1N_H)}(h_2N_H) = \alpha^*_{t(h_1)N_G}(h_2N_H) = \alpha_{t(h_1)}(h_2)N_H = h_1h_2h_1^{-1}N_H = h_1N_Hh_2N_Hh_1^{-1}N_H$.

Thus, from the above, the quotient crossed module is, in fact, a crossed module. □

Example 2.2.3. Let $f : \mathbb{X} \to \mathbb{X}^*$ be a morphism of crossed modules and consider the kernel crossed submodule constructed from $f = (\gamma, \delta)$. Then $\mathbb{X}/\text{Ker}\mathbb{X} = (H/\text{Ker}\delta, G/\text{Ker}\gamma, \bar{t}, \overline{\alpha})$ forms a quotient crossed module, where $\bar{t}(h\text{Ker}\delta) = t(h)\text{Ker}\gamma$ and $\overline{\alpha}_{g\text{Ker}\gamma}(h\text{Ker}\delta) = \alpha_g(h)\text{Ker}\delta$.

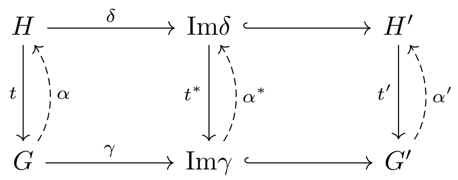
Lemma 4. Let $\mathbb{X} = (H, G, t, \alpha)$ and $\mathbb{X}/\mathbb{X}' = (H/H', G/G', \bar{t}, \overline{\alpha})$ be a quotient crossed module, noting that $\mathbb{X}'$ is a normal crossed submodule of $\mathbb{X}$. Then there exists a morphism of crossed modules $f : \mathbb{X} \to \mathbb{X}/\mathbb{X}'$.

Proof. Let $f = (\gamma, \delta)$ where $\gamma : G \to G/G'$ is defined by $\gamma(g) = gG'$, and $\delta : H \to H/H'$ is defined by $\delta(h) = hH'$. It is true that γ is a homomorphism because $\gamma(g_1g_2) = g_1g_2G' = g_1G'g_2G' = \gamma(g_1)\gamma(g_2)$, and same logic goes with δ. Next, we can see that $\gamma(t(h)) = t(h)G' = \bar{t}(hH') = \bar{t}(\delta(h))$. Finally for the action, we can show that $\delta(\alpha_g(h)) = \alpha_g(h)H' = \overline{\alpha}_{gG'}(hH') = \overline{\alpha}_{\gamma(g)}(\delta(h))$. □

2.2.2 Isomorphism theorems for strict 2-groups

Recall that the first isomorphism theorem for ordinary groups says the following. Let $\varphi : G \to \overline{G}$ be a group homomorphism. Then $G/\text{Ker}\varphi \approx \varphi(G)$. In order to discover an analogue of this for crossed modules, we must first define the notion of a image crossed module.

Definition 2.2.6. Let $(\gamma, \delta) : (H, G, t, \alpha) \to (H', G', t', \alpha')$ be a morphism of crossed modules. We define the image crossed module $(\text{Im}\delta, \text{Im}\gamma, t^*, \alpha^*)$ to be the sub-crossed module of (H', G', t', α') where the following diagram commutes.



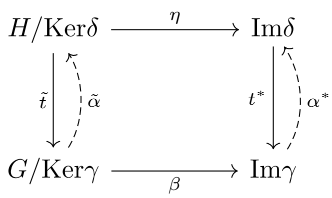
Here we define $t^* := t'|_{\text{Im}\delta} : \text{Im}\delta \to \text{Im}\gamma$ and $\alpha^* := \alpha'|_{\text{Im}\gamma \times \text{Im}\delta} \to \text{Im}\delta$.

We first must show that this is indeed a sub-crossed module of (H', G', t', α').

Proposition 10. $(\text{Im}\delta, \text{Im}\gamma, t^*, \alpha^*)$ is a subcrossed module of (H', G', t', α').

Proof. Clearly we have that $\text{Im}\delta$ and $\text{Im}\gamma$ are subgroups of H' and G' respectively. Now let $x \in \text{Im}\delta$. We must show $t^*(x) \in \text{Im}\gamma$. Indeed, if $x \in \text{Im}\delta$, then there exists $h \in H$ such that $x = \delta(h)$. Then $t^*(x) = t'(x) = t'(\delta(h)) = \gamma(t(h)) \in \text{Im}\gamma$. It remains to show that for $x \in \text{Im}\delta, y \in \text{Im}\gamma, \alpha^*_y(x) = \alpha'_y(x) \in \text{Im}\delta$. □

Theorem 1. Let $(\gamma, \delta) : (G, H, t, \alpha) \to (G', H', t', \alpha')$ be a morphism of crossed modules. Then there exists an isomorphism of crossed modules $(\eta, \beta) : (H/\text{Ker}\delta, G/\text{Ker}\gamma, \tilde{t}, \tilde{\alpha}) \to (\text{Im}\delta, \text{Im}\gamma, t^*, \alpha^*)$. In particular, the maps are given by $\eta(h\text{Ker}\delta) = \delta(h)$ and $\beta(g\text{Ker}\gamma) = \gamma(g)$.



Proof. By our preliminary results, we already know that $(H/\text{Ker}\delta, G/\text{Ker}\gamma, \tilde{t}, \tilde{\alpha})$ and $(\text{Im}\delta, \text{Im}\gamma, t^*, \alpha^*)$ are crossed modules. Moreover, we know that since η, β are group isomorphisms, if (η, β) is a morphism of crossed modules then it is automatically an isomorphism. Hence it remains to show that (η, β) is a morphism of crossed modules. Indeed, we have

$$\beta(\tilde{t}(h\text{Ker}\delta)) = \beta(t(h)\text{Ker}\gamma) = \gamma(t(h)) = t(\delta(h)) = t^*(\delta(h)) = t^*(\eta(h\text{Ker}\delta))$$

and

$$\eta(\tilde{\alpha}_{g\text{Ker}\gamma}(h\text{Ker}\delta)) = \eta(\alpha_g(h)\text{Ker}\delta) = \delta(\alpha_g(h)) = \alpha'_{\gamma(g)}(h) = \alpha^*_{\gamma(g)}(h) = \alpha^*_{\beta(g\text{Ker}\gamma)}(h)$$

as required. □

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