Exchange Paradoxes

Bemnet Bekele, James O'Hanlon, Tiffany Sun, Faculty Mentor: Douglas Eckley

Mason Experimental Geometry Lab

George Mason University

April 26th, 2024

Introduction

This semester, our research efforts largely focused on two different but related exchange paradoxes, the Two-Envelope Problem, and Siegel's Paradox. We looked to build upon the work of others and simulate thousands of trades in both problems to gain a better understanding of the stochastic probabilities underlying the problems.

Kelly Criterion

The Kelly Criterion is a formula that is used to experience optimal long-term financial growth.

$$\frac{p}{a} - \frac{(1-p)}{b}$$

Where p is the implied probability, a is the outcome of a win, and b is the outcome of a loss.

This semester, we focused on these two gambling games, and in each, the expected return is 25% when using the Kelly Criterion. In both the two-envelope problem, and the simplest version of Siegel's Paradox, we can set $p = .5$, $a = .5$, and $b = 2$. The formula essentially tells us that a player should bet half of their wealth every time for long-term gain.

Two Envelopes Paradox

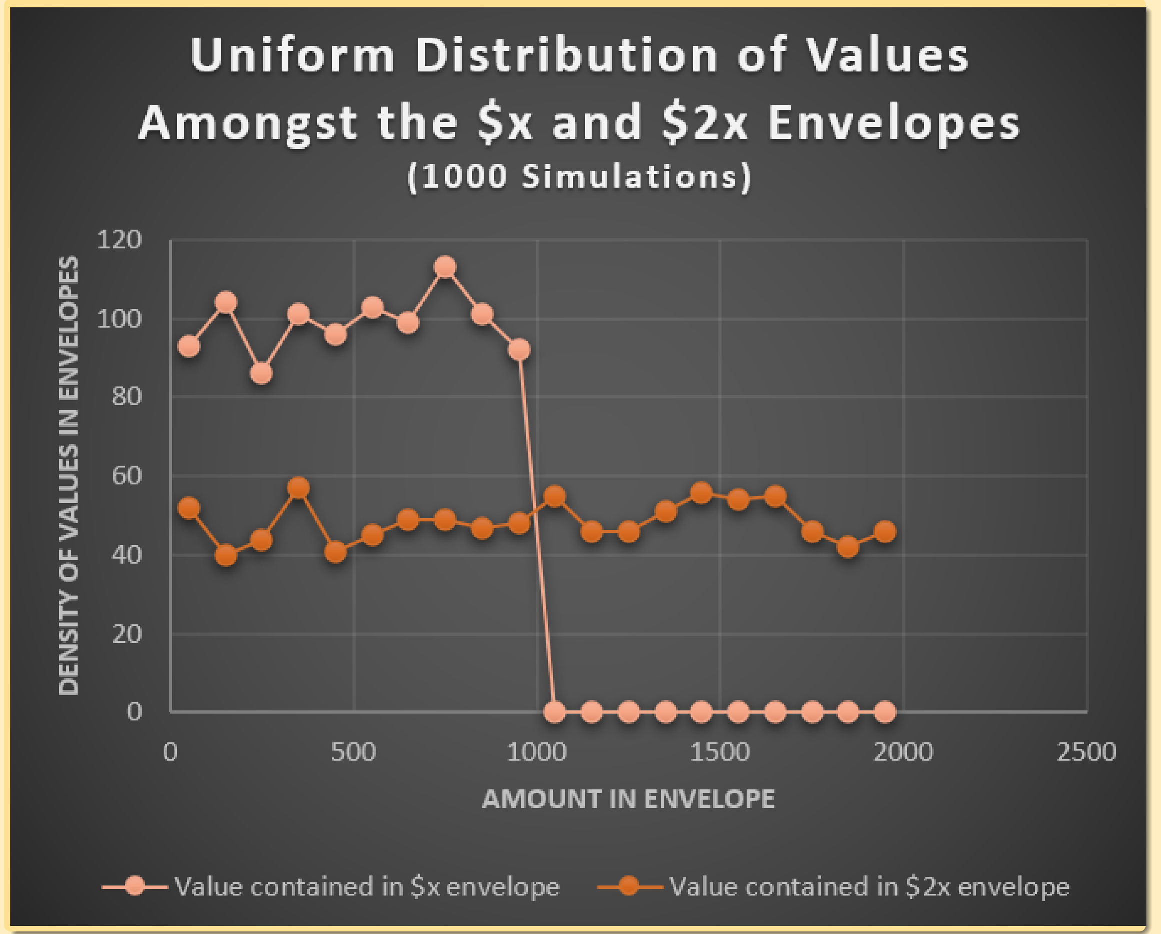
The "Two Envelope Paradox" is as follows: A host fills one envelope with \$x, and another with \$2x. The player chooses one of them and will win whatever amount it contains. The player will then open the envelope and find some arbitrary amount of money inside, let's say \$20. The player is then asked if he/she would like to trade. A probability-minded player might say "yes", reasoning that the other envelope is either \$40 or \$10, and therefore an expected gain occurs with the trade. However, it also stands to reason that because you do not know if the amount you are holding is worth \$2x or \$x, you stand no gain through trading, here in lies the paradox.

Siegel's Paradox

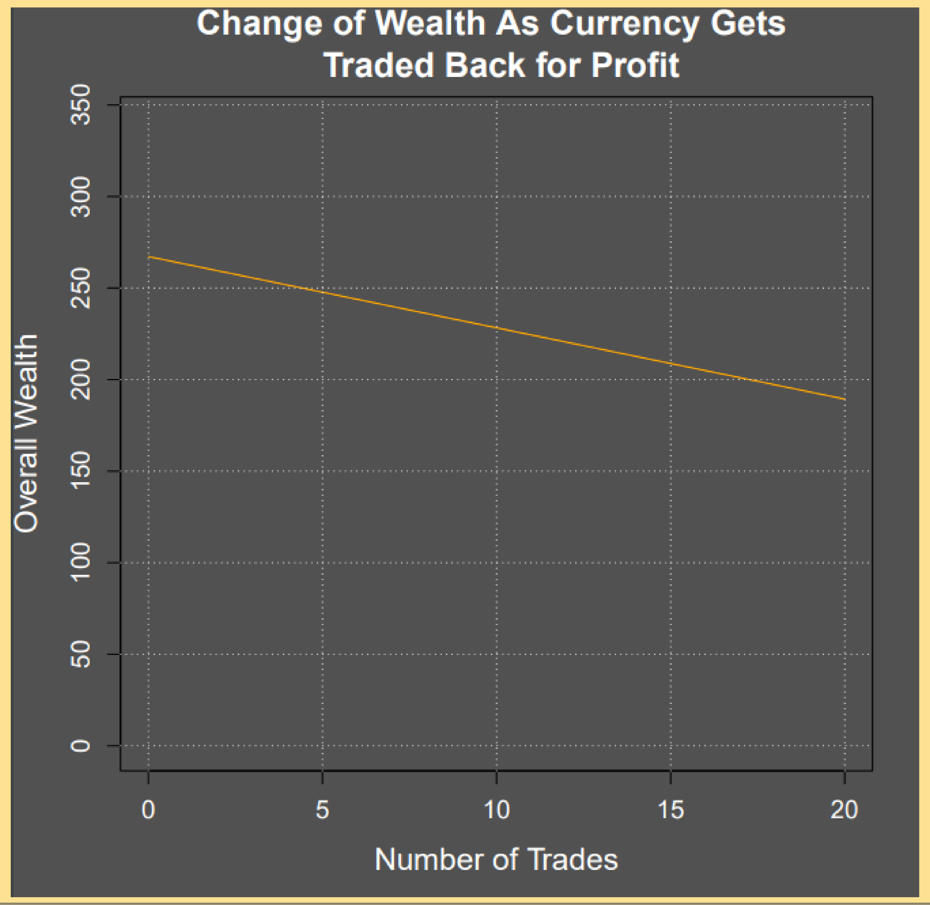
"Siegel's Paradox" is a problem involving foreign currencies and exchange rates. Suppose you had two countries, The U.S. and Japan, who currently traded at a 1 : 1 exchange rate. Now suppose that the exchange rate has a fair chance of going to either 2 : 1 or 1 : 2 next year. A quick calculation of expected value shows that somehow both countries stand to gain by investing in the other currency, but how can this be? This problem can further be expanded to contain different exchange rates, and more countries/players.

Two Envelopes Model

Our research indicates that in the simplest version of the envelope problem, it would never be advantageous to a player to switch envelopes. Because an understanding of whether the value a player is holding is \$2x or \$x is never learned. We further pursued this problem by creating variants and exploring different probability distributions. Suppose that the value of the envelope containing \$x is limited to be at maximum \$1000, and the value of x is randomly distributed. If the player opens an envelope to find less than $2E[X]$, where $E[X]$ is the expected value of x based on the distribution, then the player would find an advantage in trading, as it is more likely he is holding the envelope containing \$x. To the right is a graph of 1000 simulations where \$x is uniformly distributed.



Siegel's Paradox Model



We built a stochastic model of foreign currency trading to model Siegel's Paradox. We initially set the world's overall portfolio with 50% of the value in dollars and 50% yen. After we execute some trades, we notice gains exceed losses, However, when the model attempts to trade that wealth back into its native currency, any profit that was seemingly made is lost. The following equations were used to help build the model:

$$y = x \cdot r_{.5}$$

$$\frac{r_1}{r_0} = (1 + k\sqrt{xy})^{\pm 1}$$

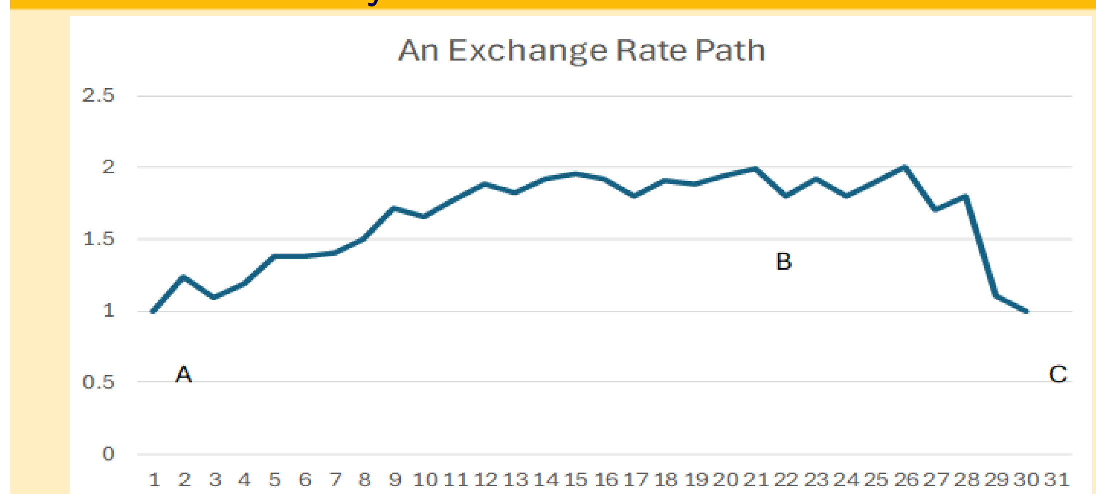
$$r_{.5} = \sqrt{r_0 \cdot r_1}$$

k is constant, x is dollars buying yen, y is yen buying dollars, r_0 is yen:dollar rate, r_1 is dollar:yen rate, and $r_{.5}$ is the rate at which a trade is executed.

Siegel continued..

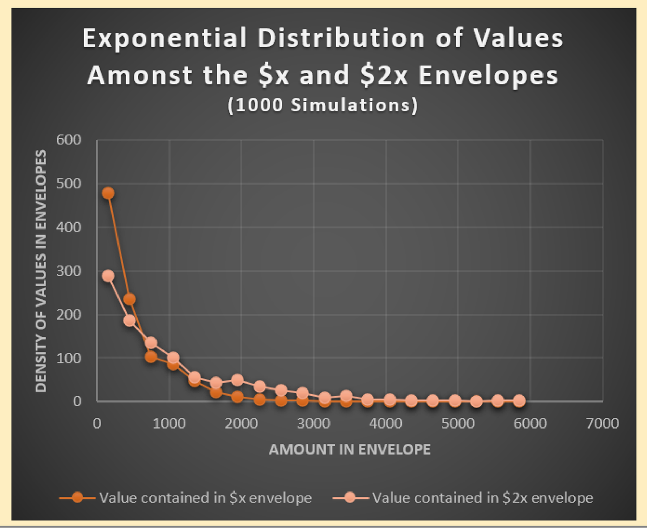
The graph to the left shows our simulation of Siegel's Paradox. A represents a player's starting position. B represents the players' apparent wealth after a certain amount of trades. And C represents the trade a player must make to convert the wealth back into its domestic currency. As is indicated by the graph, the profit is mostly lost when trading back.

Wealth after many trades:



Exponentially Distributed Envelopes

The following graph depicts an exponential distribution of envelope values, rather than uniform.



Conclusions/Future Work

We have concluded that the Two Envelopes problem can be properly solved and understood when the density of values in each respective envelope is taken into consideration. Furthermore, the problem can be extended into a variety of different stochastic distributions and values to simulate more complex alternates to the original problem. We have also concluded that in Siegel's paradox, all players do not have an expected gain by investing in a different currency. After the exchange rate changes, it would be difficult or impossible to trade back into a profit, thus the expected value from switching is illusionary for large players. However, with a large enough amount of players and overall bankroll in the economy, it may be possible for a small player to make small gains by switching currencies.

Acknowledgments

We extend our gratitude to Anton Luyanenko and the MEGL management team for allowing us the resources and opportunity to conduct our research. As well as thanking Dr. Douglas Eckley for all his advice and wisdom throughout the semester.

References

1. McDonnell, M. D. & Abbott, D. Randomized Switching in the Two-Envelope Problem. *Proceedings: Mathematical, Physical and Engineering Sciences*, 465(2111), 2009
2. Siegel, Jeremy J. Risk, Interest Rate and the Forward Market. *Quarterly Journal of Economics* 86:303–309, 1972
3. Chu, K.H. Solution to the Siegel Paradox. *Open Econ Rev*, 16:399–405, 2005
4. Nalebuff. Puzzles The Other Person's Envelope is Always Greener. *Journal of Economic Perspectives*, 3(1):171-181, 1989
5. Yi, Byeong-Uk. The Two-envelope Paradox With No Probability, 2009