Exchange Paradoxes

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About Our Research

Research Goal:

This semester our team set out with the goals to solve the Two-Envelope Problem and Siegel's Paradox using data simulation to explain the fallacy in both problems.

Our research methods involved looking at the previous work on these problems and then creating simulations in both R and Python to shed light on the solutions. With our simulation models, we were able to accurately show why there is no actual gain in trading even though there appears to be.

Kelly Criterion

The Kelly Criterion is a formula used to maximize the long-term geometric rate of return from a casino game. The result tells you the proportion of your wealth that should be wagered on a game.

$$\frac{p}{a} - \frac{(1-p)}{b}$$

Where p is the implied probability, a is the outcome of a win, and b is the outcome of a loss.

Kelly Criterion

We applied the Kelly Criterion to two different paradoxes involving wealth and probability. The formula will only tell you to gamble if you have an expected gain. In our two problems, it appears as though there exists an expected gain, and Kelly tells us to invest 75% of our wealth each time we play. Yet, in both cases, the apparent gains have paradoxical aspects. This led us to investigate both.

The Two-Envelopes Problem

History

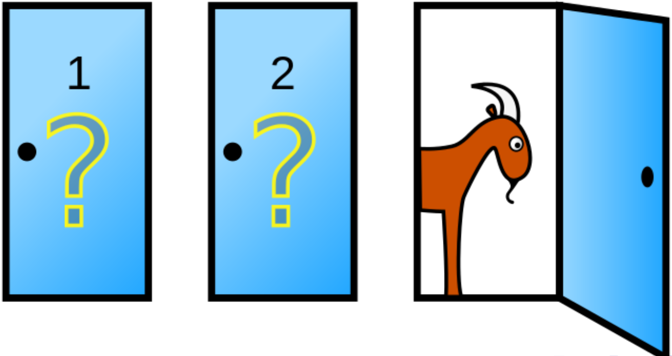
The exact date of creation for the Two-Envelopes Problem is disputed, however, it dates back to at least 1953 when described by Belgian mathematician Maurice Kraitchik. Since then, the solution to the issue has been largely disputed by both mathematicians and economists alike.

The Two-Envelope Paradox

The problem follows as such: Suppose a host fills two identical envelopes, one with \$x and the other with \$2x. Assuming the probability is fair, a player is asked to pick an envelope, at this point a player has no reason to switch envelopes. However, the host then allows the player to look inside the envelope to reveal the money. Let's say that the amount inside is \$20. That means that the other envelope contains either \$10 or \$40, so when taking the expected value of the other envelope, we can see that it would be in the player's favor to switch. But this doesn't make sense!

The Monty-Hall Problem

This problem may seem reminiscent of another somewhat famous result in probability, the Monty-Hall problem. Where a contestant has 3-doors, behind one of which is a car, and the other 2 are goats. A player selects a door, and then the host reveals the goat behind one of the doors the player did not select. The question then is, should you switch to the other door or stick with your own?



The Two-Envelope Problem

Our first approach to this problem involved building simulation models in Python to represent the Two-Envelopes situation and then running the simulation thousands of times to reveal a long-term trend. Our simulations showed us that when the envelopes are filled before the game takes place, then there is no monetary advantage to switching envelopes. However, if the other envelope is determined by a random variable, such that it has a 50% chance of doubling and a 50% chance of halving, then there would exist an advantage to switching. We believe that much debate surrounding the topic revolves around which situation the Two-Envelopes problem resembles.

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Player won $750120165074.0 when not switching
Player won $750807240757.0 when switching
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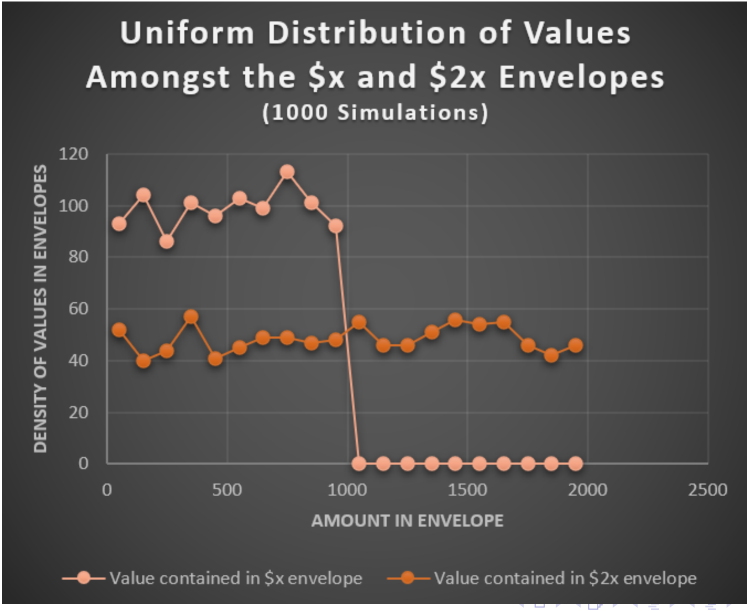
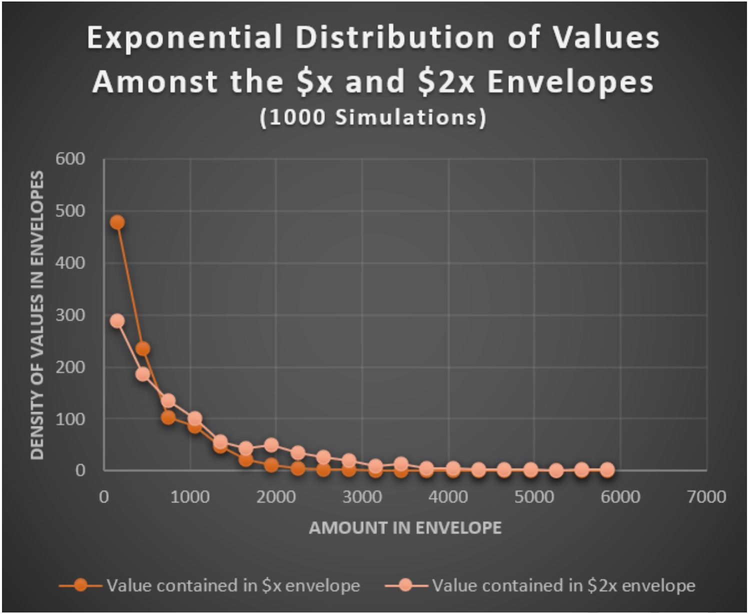
The Two-Envelopes Problem

We then expanded the problem to include more complex random distributions. For example, imagine if you knew that the smaller of the two envelopes, would be uniformly distributed between $[1, 1000]$ dollars. According to our simulations, it would be advantageous to switch anytime you encounter a number that is less than twice the expected value of the smaller envelope. Which intuitively seems to make some sense.

Uniform Distribution



Exponential Distribution



Siegel's Paradox

Siegel's Paradox was first proposed by economist Jeremy Siegel in 1972. It focuses on the expectation of profit between foreign currencies and exchange rates. According to the situation described by Siegel, it would seem that all parties stand to gain by investing in foreign currency, even though logically it seems to make no sense.

Siegel's Paradox

Siegel's Paradox

"Siegel's Paradox" is a problem involving foreign currencies and exchange rates. Suppose you had two countries, The U.S. and Japan, who currently traded at a $1:1$ exchange rate. Now suppose that the exchange rate has a fair chance of going to either $2:1$ or $1:2$ next year. A quick calculation of expected value shows that somehow both countries stand to gain by investing in the other currency, but how can this be? This problem can further be expanded to contain different exchange rates, and more countries/players.

The problem is similar to the Two-Envelopes Problem in the sense that a trade seems to lead to profit, even though trading offers no advantage. We developed an equation that seemed to sensibly model an exchange rate model, where exchange rates randomly vary. With this model, we were able to show that when a trade is made, both countries would appear to have increased their wealth, however when they attempt to trade back into domestic currency, most, if not all, of the profit disappears.

Modelling Siegel's Paradox

Modelling Equation

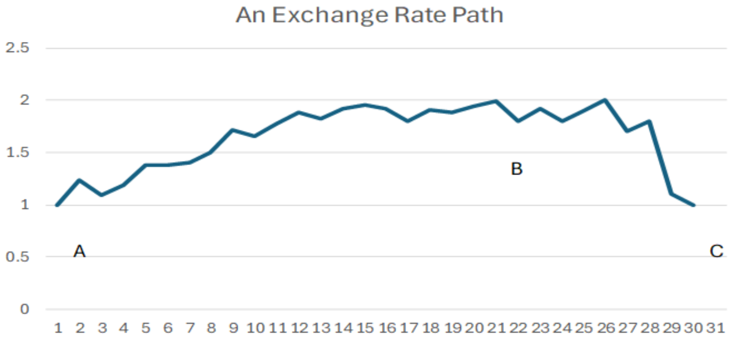
$$y = x \cdot r_{.5}$$

$$\frac{r_1}{r_0} = (1 + k\sqrt{xy})^{\pm 1}$$

$$r_{.5} = \sqrt{r_0 \cdot r_1}$$

k is constant, x is dollars buying yen, y is yen buying dollars, r_0 is yen:dollar rate, r_1 is dollar:yen rate, and $r_{.5}$ is the rate at which a trade is executed.

Modelling Siegel's Paradox



As can be seen by the graph, a country does tend to generally increase its wealth by incrementally trading into a foreign currency. However, because no wealth is actually generated by the market, when trading back into domestic currency the rate becomes very unfavorable and all apparent gains are lost. Another way of looking at it is that the profit made through investing in foreign currency directly affects the value of your own currency, so now real changes in wealth are made.

Conclusion

The Two-Envelopes problem as well as Siegel's paradox are somewhat interrelated problems regarding exchange rates and expectations. We have concluded that in the case of the envelope problem, there is no actual advantage in switching and the other envelope cannot be considered as a 50/50 chance of halving or doubling. When further restrictions are placed on the envelope problem, then there does seem to be a point at which switching is a viable option, given you have a certain amount of information. In Siegel's Paradox as well, we conclude that a player controlling a large amount of wealth has no advantage in trading their currency, however a smaller player may hold an advantage for a small amount of money.

Future Work

In our research, there are still areas in our work that we could go back on and be more rigorous in explaining the processes going on. The Envelope Problem, in particular, there are many more possible variants of the game that have not been explored. We could also hope to construct a much larger and more efficient simulation of Siegel's paradox, to test our hypothesis that a small enough player could make a solid financial gain through randomly distributed exchange rates.

Acknowledgements

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