Extended Archimedes Principle for Floating Objects Experiencing Surface Tension

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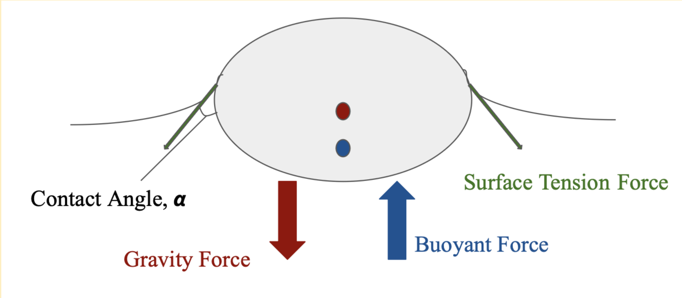
Introduction

An extended application of Archimedes Principle is utilized to incorporate the force of surface tension into the force balance equation by computationally describing object and meniscus interactions. The equilibrium points of this force balance equation give us the stable floating configurations for 3D printed objects with a uniform cross-sectional area.

Forces Acting on a Floating Object

The orientation of a stationary floating object may be characterized by the balance of gravitational force, buoyant force, and surface tension force.



Mathematically, this may be written:

$$0 = \vec{F_g} + \vec{F_p} + \vec{F_T}$$

With forces defined below:

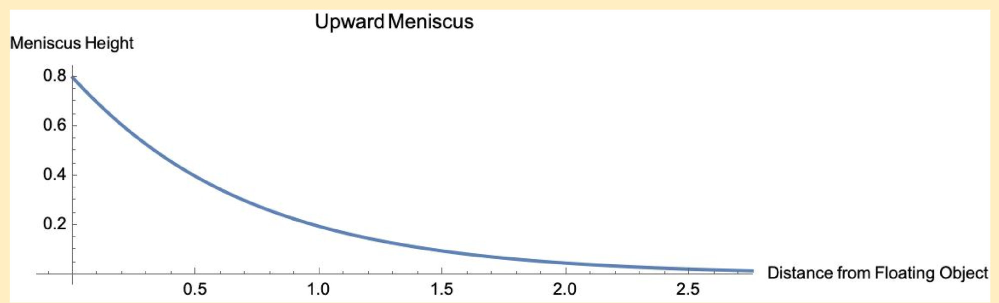
$$\vec{F_g} = M_{obj}\vec{g}$$

$$\vec{F_p} = \ell\rho_f g\{\frac{1}{2}({y_R}^2 - {y_L}^2), A_{sub} - A_{\text{meniscus}}\}$$

$$\vec{F_T} = \ell\gamma_L\vec{t_L} + \ell\gamma_R\vec{t_R}$$

Contact Angle and Shape of the Meniscus

Equilibrium contact angle, α, is intrinsic to a system of a solid intersecting a phase interface at 90°. That is, contact angle is measured such that the object is essentially a vertical wall. α is determined from reliable experimental data. The expected direction of a meniscus can therefore be predicted from α relative to 90°.



Bond Number and Relevance of Meniscus

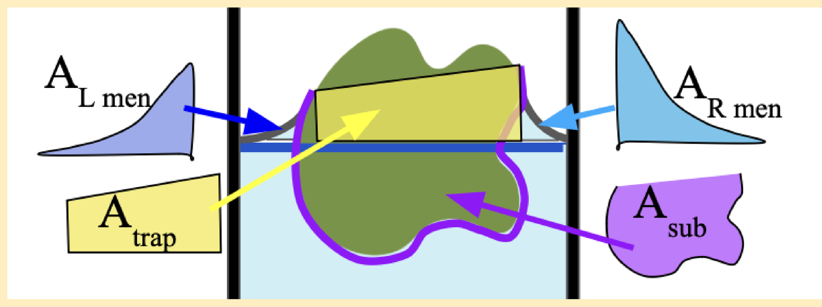
The Bond Number, B_0, is a dimensionless parameter used to compare the magnitude of the gravitational and surface tension forces experienced by a floating object at a phase interface.

$$B_0 = \frac{\Delta\rho\ell^2 g}{\gamma}$$

where $\Delta\rho = \rho_{fluid}\text{-}\rho_{object}$, ℓ is the length scale of the object, g is acceleration due to gravity, and γ is the surface tension constant. Small objects will have a smaller bond number and experience higher surface tension affects in comparison to gravitational force.

Extended Archimedes Principle

The buoyant force acting on an object floating in a fluid is equal and opposite to the force of gravity acting on the object. If the bond number is sufficiently large, then surface tension becomes negligible. Otherwise, the fluid displaced from the waterline significantly contributes to the buoyant force and the area of the meniscus must be accounted for in A_{sub}.



$$f_A(H, \Delta y_L, \Delta y_R) = -\rho_{obj}A_{obj} + \rho_f(A_{sub} - A_{trap} - A_{Lmen} - A_{Rmen}) = 0$$

$$f_B(H, \Delta y_L, \Delta y_R) = -(\Delta y_R)^2 + 2d^2(1 - \sin(\alpha_R)) = 0$$

$$f_C(H, \Delta y_L, \Delta y_R) = -(\Delta y_L)^2 + 2d^2(1 - \sin(\alpha_L)) = 0$$

The above functions describe the relationships of the extended Archimedes Principle, f_A, and the meniscus geometry with respect to fluid properties, f_B and f_C.

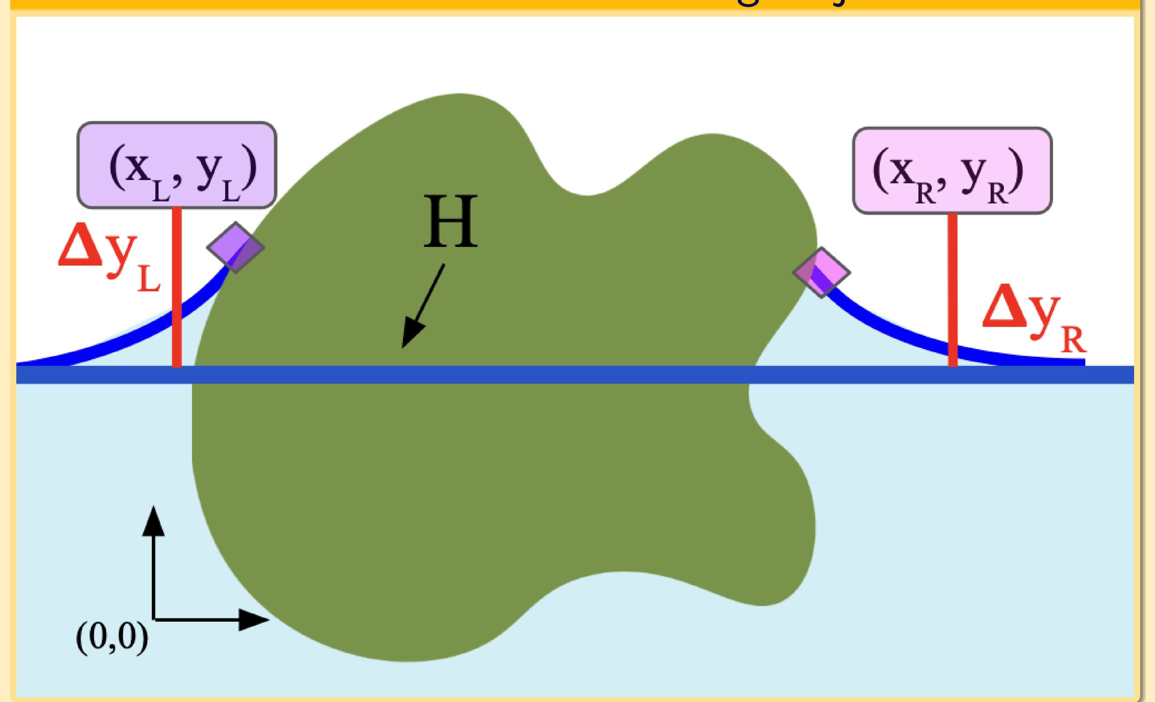
Computational Interpretation

The geometry of an object can be defined by a set of X-Y coordinate points that outline a polygon. The three functions that describe the relationships of the extended Archimedes Principle form a system of equations with three inputs - H, Δy_L, Δy_R - that describe a stable floating configuration when minimized. H defines the height of the waterline above the X-axis and the Δy_L and Δy_R describe the meniscus contact points relative to H. Newton's Method is used to solve this system of equations because $f = (f_A, f_B, f_C)$ is from $R^3 \to R^3$.

$$\Delta y_L = (H - y_L)$$

$$\Delta y_R = (H - y_R)$$

3 Dimensions of Problem on Floating Object

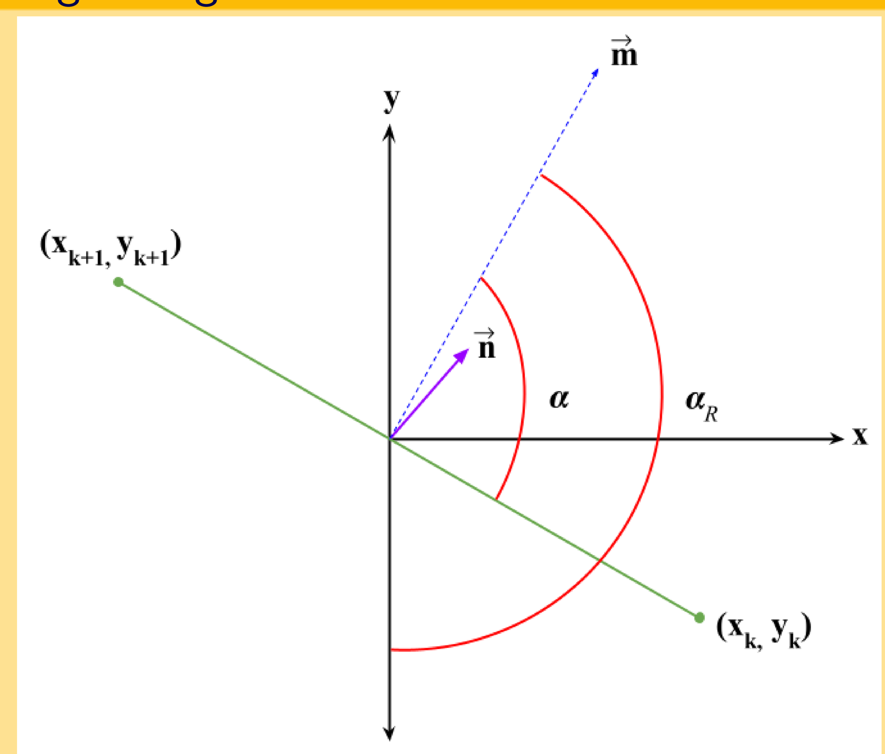


Newton's Method in Higher Dimensions

$$x_{k+1} = x_k - (DF(x_k))^{-1}F(x_k)$$

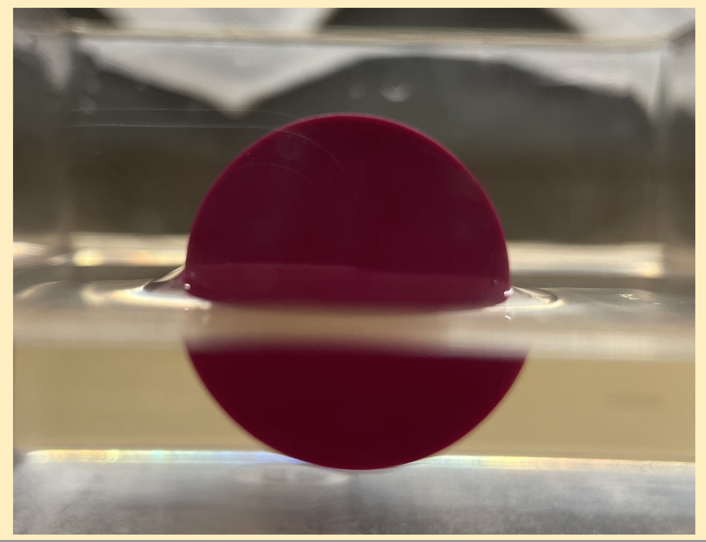
To solve this problem, which is given by 3 dimensions, $x = \{H, \Delta y_L, \Delta y_R\}$ and 3 equations, $F = \{f_A, f_B, f_C\}$, Newton's Method in higher dimensions is used to find a numerical approximation. Newton's Method requires an initial approximate solution to iterate upon. Here, initial approximations of the 3 unknowns will be derived from previous code that does not account for surface tension.

Left and Right Angles



Case Study: Floating Cylinders

Two upward facing menisci can be observed on the left and right sides of the cylinder. A cylinder is observed in this case study because it is a convex shape, which is a necessary condition for the current model. The cylinder also has a circular cross section, meaning that all orientations have equal energy. The potential energy for the current surface tension model has not yet been adapted to describe the potential energy minima.



Future Work

1. Implement Newton's Method framework to analyze convex shapes
2. Build potential energy model for extended Archimedes Principle
3. Compare model's predictions to experimental results
4. Adapt model for non-convex shapes
5. Apply these findings to 3D-printed objects that float on top of a liquid without breaking its surface, as seen in certain insect species

Acknowledgments

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References

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