Finite 2-Groups

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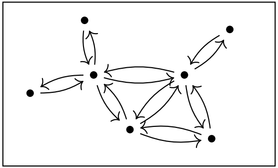
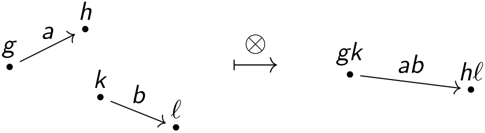
2-Group:



$(\mathcal{G}, \otimes)$



Definition

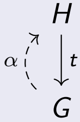
A (strict) **2-group** is a category (i.e. a collection of objects and morphisms) with a binary operation ($\otimes$) such that:

- The operation is closed and associative.
- There exists an identity object.
- All objects and morphisms are invertible under $\otimes$

Background: Crossed Modules

Definition

A **crossed module** $\mathbb{X} = (G, H, t, \alpha)$ consists of two groups G and H, a homomorphism $t : H \to G$, and a group action $\alpha : G \to \mathrm{Aut}H$.



Satisfying equivariance:

$$t(\alpha_g h) = gt(h)g^{-1}$$

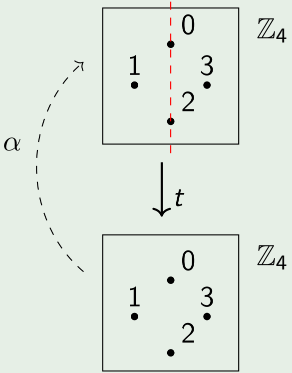
and the Peiffer identity:

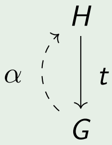
$$\alpha_{t(h)} h' = hh'h^{-1}$$

Background: Smallest Nontrivial Crossed Module

Example

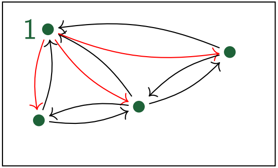
Consider the crossed module $\mathbb{X} = (\mathbb{Z}_4, \mathbb{Z}_4, t, \alpha)$, letting $t(1) = 2$ and $\alpha_1(1) = \alpha_3(1) = 3$, and $\alpha_1(3) = \alpha_3(3) = 1$.
0 and 2 act trivially: $\alpha_0(h) = \alpha_2(h) = h$.



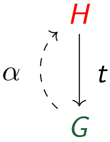


Background: 2-Groups and Crossed Modules Induce Each Other

We can construct a crossed module from a 2-group: H represents the unit-source morphisms, and G represents the objects in a 2-group.



$(\mathcal{G}, \otimes)$



Homomorphism $t : H \to G$ is represented by the targets of the unit-source morphisms, and α is represented by conjugation $\mathrm{id}_g \otimes h \otimes \mathrm{id}_{g^{-1}}$.

Similar argument holds to show the construction of a 2-group from a crossed module, with α as the semidirect product $(h, g) \otimes (h', g') = (h\alpha_g(h'), gg')$.

Current Results

- *First Isomorphism Theorem for Crossed Modules*
  - Verified definition of a homomorphism and isomorphism of crossed modules
  - Verified definition of kernel, image, normal and quotient crossed modules
- Progress on *Fundamental Theorem of Finite Abelian Crossed Modules*
  - Asserted definition of a cyclic crossed module, and a "direct sum of cyclic crossed modules"
  - Current strategies of furthering progress on such a theorem.

Crossed Submodules, Normal Crossed Submodules

Definition

A **crossed submodule** $\mathbb{X}^* = (G^*, H^*, t^*, \alpha^*)$ of a crossed module $\mathbb{X} = (G, H, t, \alpha)$ satisfies the following conditions:
G^* is a subgroup of G, H^* is a subgroup of H, t^* is the restriction of t on H^*, and α^* is the restriction of α on G^*, H^*.

Definition

A crossed submodule $\mathbb{X}^* = (G^*, H^*, t^*, \alpha^*)$ is **normal** if it satifies these conditions:

- G^* is a normal subgroup of G.
- For $g \in G, h' \in H^*$, $\alpha_g(h') \in H^*$.
- For $h \in H, g' \in G^*$, $\alpha_{g'}(h)h^{-1} \in H^*$

Remark: H^* being a normal subgroup of H is guaranteed by the Peiffer identity.

Quotient Crossed Module

Definition

Given a crossed module $\mathbb{X}$ and a crossed submodule $\mathbb{X}^*$, a **quotient crossed module** $\mathbb{X}/\mathbb{X}^* = (G/G^*, H/H^*, \tilde{t}, \tilde{\alpha})$ can be constructed such that $\tilde{t}(hH^*) = t(h)G^*$ and $\tilde{\alpha}_{gG^*}hH^* = \alpha_g(h)H^*$.

Theorem

The construction above is a crossed module.

In order to prove this, we must show that:

- $\tilde{t}, \tilde{\alpha}$ are well-defined
- $\tilde{t}$ is a group homomorphism, $\tilde{\alpha}$ is a group action
- $\tilde{t}, \tilde{\alpha}$ satisfy equivariance and the Peiffer identity.

Recall we define $\tilde{t}$ by $\tilde{t}(hH^*) = t(h)G^*$. To show this is a well defined map, we show that $\tilde{t}$ is independent of the choice of h.

Proof.

$\tilde{t}$ **is well-defined**. Let $h_1 H^* = h_2 H^*$. We want to show $t(h_1)G^* = t(h_2)G^*$. Recall that if $h_1 H^* = h_2 H^*$, there exists $n \in H^*$ such that $h_1 = h_2 n$.
Thus, $t(h_1)G^* = t(h_2 n) = t(h_2)t(n)G^* = t(h_2)G^*$.

□

Proof.

$\tilde{\alpha}$ **is well-defined**. Let $h_1H^* = h_2H^*$ and $g_1G^* = g_2G^*$. We want to show $\alpha_{g_1}(h_1)H^* = \alpha_{g_2}(h_2)H^*$. There exists $n \in H^*, n' \in G^*$ such that $h_1 = h_2n, g_1 = g_2n'$.
Thus, $\alpha_{g_1}(h_1)H^* = \alpha_{g_2n'}(h_2n) = \alpha_{g_2n'}(h_2)\alpha_{g_2n'}(n)H^* = \alpha_{g_2n'}(h_2)H^*$ since $\alpha_{g_2n'}(n) \in H^*$ by $\mathbb{X}^*$ being a normal crossed submodule.

$$\begin{aligned}
\alpha_{g_2n'}(h_2)H^* &= \alpha_{g_2}(\alpha_{n'}(h_2))H^* \\
&= \alpha_{g_2}(\alpha_{n'}(h_2)h_2^{-1}h_2)H^* \\
&= \alpha_{g_2}(\alpha_{n'}(h_2)h_2^{-1})\alpha_{g_2}(h_2)H^* \\
&= \alpha_{g_2}(\alpha_{n'}(h_2)h_2^{-1})H^*\alpha_{g_2}(h_2) \\
&= H^*\alpha_{g_2}(h_2) = \alpha_{g_2}(h_2)H^*
\end{aligned}$$

. □

Quotient Crossed Modules are Crossed Modules (pt. 3)

Proof.

$\tilde{t}$ **is a group homomorphism**. $\tilde{t}(h_1H^*h_2H^*) = \tilde{t}(h_1h_2H^*) = t(h_1h_2)G^* = t(h_1)t(h_2)G^* = t(h_1)G^*t(h_2)G^* = \tilde{t}(h_1H^*)\tilde{t}(h_2H^*)$.

$\tilde{\alpha}$ **is a group action**.

- $\tilde{\alpha}_{e_G G^*}(hH^*) = \alpha_{e_G}(h)H^* = hH^*$.
- $\tilde{\alpha}_{g_1 G^*}(\tilde{\alpha}_{g_2 G^*}(hH^*)) = \tilde{\alpha}_{g_1 G^*}(\alpha_{g_2}(h)H^*) = \alpha_{g_1}(\alpha_{g_2}(h))H^* = \alpha_{g_1 g_2}(h)H^* = \tilde{\alpha}_{g_1 g_2 G^*}(hH^*) = \tilde{\alpha}_{g_1 G^* g_2 G^*}(hH^*)$.
- $\tilde{\alpha}_{gG^*}(h_1H^*h_2H^*) = \tilde{\alpha}_{gG^*}(h_1h_2H^*) = \alpha_g(h_1h_2)H^* = \alpha_g(h_1)\alpha_g(h_2)H^* = \alpha_g(h_1)H^*\alpha_g(h_2)H^* = \tilde{\alpha}_g(h_1H^*)\tilde{\alpha}_g(h_2H^*)$

□

Quotient Crossed Modules are Crossed Modules (pt. 4)

Recall that in order for (H, G, t, α) to form a crossed module, t and α must satisfy equivarience

$$t(\alpha_g h) = gt(h)g^{-1}$$

and the Peiffer identity:

$$\alpha_{t(h)}h' = hh'h^{-1}$$

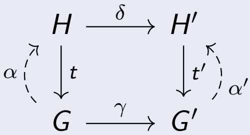
Proof.

$\tilde{t}, \tilde{\alpha}$ **satisfy equivariance and the Peiffer identity**.
Equivariance: $\tilde{t}(\tilde{\alpha}_{gG^*}(hH^*)) = \tilde{t}(\alpha_g(h)H^*) = t(\alpha_g(h))G^* = gt(h)g^{-1}G^* = gG^*t(h)G^*g^{-1}G^* = gG^*\tilde{t}(hH^*)g^{-1}G^*$
Peiffer Identity:
$\tilde{\alpha}_{\tilde{t}(hH^*)}(h'H^*) = \alpha_{t(h)}(h')H^* = hh'h^{-1}H^* = hH^*h'H^*h^{-1}H^*$ □

Morphisms of Crossed Modules

Definition

A **morphism of crossed modules** $f = (\delta, \gamma) : \mathbb{X} \to \mathbb{X}'$, with $\mathbb{X} = (G, H, t, \alpha)$ and $\mathbb{X} = (G', H', t', \alpha')$, refers to the diagram below:



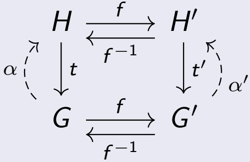
The diagram satisfies three conditions:

- γ, δ are group homomorphisms
- $t' \circ \delta = \gamma \circ t$
- $\delta(\alpha_g(h)) = \alpha'_{\gamma(g)}(\delta(h))$.

Isomorphisms of Crossed Modules

Definition

Let $f = (\delta, \gamma) : (H, G, t, \alpha) \to (H', G', t', \alpha')$ be a morphism of crossed modules. Then f is an *isomorphism* of crossed modules if there exists another morphism $f^{-1} : (H', G', t', \alpha') \to (H, G, t, \alpha)$ so that
$f^{-1} \circ f = \mathrm{id} : (H, G, t, \alpha) \to (H, G, t, \alpha)$ and
$f \circ f^{-1} = \mathrm{id} : (H', G', t', \alpha') \to (H', G', t', \alpha')$.



Lemma

Let (δ, γ) be a morphism of crossed modules. If δ, γ are group isomorphisms, then (δ, γ) is an isomorphism of crossed modules.

Example

Given a morphism of crossed modules $f : \mathbb{X} \to \mathbb{X}'$, the **kernel crossed submodule** $\mathrm{Ker} f = (\mathrm{Ker}\gamma, \mathrm{Ker}\delta, t^* = t|_{\mathrm{Ker}\delta}, \alpha^* = \alpha|_{\mathrm{Ker}\gamma \times \mathrm{Ker}\delta})$ is a crossed submodule of $\mathbb{X}$.

Proof.

$$\begin{array}{ccccc} \mathrm{Ker}\delta & \longrightarrow & H & \xrightarrow{\delta} & H' \\ {\scriptstyle \alpha^*} \circlearrowleft \downarrow {\scriptstyle t^*} & & {\scriptstyle \alpha} \circlearrowleft \downarrow {\scriptstyle t} & & \downarrow {\scriptstyle t'} \circlearrowright {\scriptstyle \alpha'} \\ \mathrm{Ker}\gamma & \longrightarrow & G & \xrightarrow{\gamma} & G' \end{array}$$

$\mathrm{Ker}\gamma$ and $\mathrm{Ker}\delta$ are subgroups. Since the diagram commutes, it implies that $\mathrm{Ker}\delta$ must map to $\mathrm{Ker}\gamma$, and $e_H = \alpha'_{\gamma(g)}(\delta(h)) = \delta(\alpha_g(h))$ for $g \in \mathrm{Ker}\gamma, h \in \mathrm{Ker}\delta$ implies $\alpha_g(h) \in \mathrm{Ker}\delta$. □

The Kernel Crossed Submodule (pt. 2)

Example

The **kernel crossed submodule** Ker$f$ is a normal crossed submodule.

Proof.

- Kerγ is a normal subgroup of G.
- $\alpha_g(h') \in \mathrm{Ker}\delta$ since $\delta(\alpha_g(h^*)) = \alpha'_{\gamma(g)}(\delta(h^*)) = e_{H'}$ if $h^* \in \mathrm{Ker}\delta$.
- $\delta(\alpha_{g^*}(h)) = \alpha'_{e_{G'}}(\delta(h)) = \delta(h)$, which implies $\delta(\alpha_{g^*}(h))\delta(h^{-1}) = \delta(h)\delta(h^{-1})$, and thus $\delta(\alpha_{g^*}(h)h^{-1}) = e_{H'}$ so $\alpha_{g^*}(h)h^{-1} \in \mathrm{Ker}\delta$.

□

The Image Crossed Submodule

Example

Given a morphism of crossed modules $f = (\gamma, \delta) : (H, G, t, \alpha) \to (H', G', t', \alpha')$, the **image crossed submodule** $\mathrm{Im} f = (\mathrm{Im}\gamma, \mathrm{Im}\delta, t^* = t'|_{\mathrm{Im}\delta}, \alpha^* = \alpha'|_{\mathrm{Im}\gamma \times \mathrm{Im}\delta})$ is a crossed submodule of $\mathbb{X}'$.

Proof.

$$\begin{array}{ccccc} H & \xrightarrow{\delta} & \mathrm{Im}\delta & \longrightarrow & H' \\ \alpha \ \ \downarrow t & \alpha^* & \downarrow t^* & & \downarrow t' \ \ \alpha' \\ G & \xrightarrow{\gamma} & \mathrm{Im}\gamma & \longrightarrow & G' \end{array}$$

It follows that $\mathrm{Im}\gamma$, $\mathrm{Im}\delta$ are subgroups of G', H' respectively. By the diagram commuting, $t'|_{\mathrm{Im}\delta}$ exists, and $\delta(\alpha_g(h)) = \alpha'_{\gamma(g)}(\delta(h))$ implies $\alpha'|_{\mathrm{Im}\gamma \times \mathrm{Im}\delta}$ exists. □

The First Isomorphism Theorem for Crossed Modules

Theorem

Let $(\gamma, \delta) : (G, H, t, \alpha) \to (G', H', t', \alpha')$ *be a morphism of crossed modules. Then there exists an isomorphism of crossed modules* $(\beta, \eta) : (H/Ker\delta, G/Ker\gamma, \tilde{t}, \tilde{\alpha}) \to (Im\delta, Im\gamma, t'|_{Im\delta}, \alpha'|_{Im\gamma \times Im\delta})$*. In particular, the maps are given by* $\eta(hKer\delta) = \delta(h)$ *and* $\beta(gKer\gamma) = \gamma(g)$*.*

Proof.

$$\begin{array}{ccc} H/\mathrm{Ker}\delta & \xrightarrow{\eta} & \mathrm{Im}\delta \\ \tilde{\alpha} \circlearrowright \; \big\downarrow \tilde{t} & & t'|_{\mathrm{Im}\delta} \big\downarrow \; \circlearrowleft \alpha'|_{\mathrm{Im}\gamma \times \mathrm{Im}\delta} \\ G/\mathrm{Ker}\gamma & \xrightarrow[\beta]{} & \mathrm{Im}\gamma \end{array}$$

β, η are isomorphisms from the first isomorphism theorem for groups.
$\beta(\tilde{t}(h\mathrm{Ker}\delta)) = \beta(t(h)\mathrm{Ker}\gamma) = \gamma(t(h)) = t'(\delta(h)) = t'(\eta(h\mathrm{Ker}\delta))$.
$\eta(\tilde{\alpha}_{g\mathrm{Ker}\gamma}(h\mathrm{Ker}\delta)) = \eta(\alpha_g(h)\mathrm{Ker}\delta) = \delta(\alpha_g(h)) = \alpha'_{\gamma(g)}(h) = \alpha^*_{\gamma(g)}(h) = \alpha^*_{\beta(g\mathrm{Ker}\gamma)}(h)$. Similar argument for the other direction.

□

Finite Abelian 2-Groups

Definition

A **finite abelian 2-group** is a 2-group with finitely many objects and morphisms that has its operation ($\otimes$) commutative.

Theorem

A finite abelian 2-group induces a crossed module where G, H are abelian and α is trivial: $\alpha_g h = h$ for $g \in G, h \in H$.

Proof.

If $\otimes$ is communtative for objects and morphisms in $\mathcal{G}$, then certainly G is abelian and H is abelian by construction.
Since α is conjugation ($\mathrm{id}_g \otimes h \otimes \mathrm{id}_{g^{-1}}$), then:
$\mathrm{id}_g \otimes h \otimes \mathrm{id}_{g^{-1}} = \mathrm{id}_g \otimes \mathrm{id}_{g^{-1}} \otimes h = \mathrm{id}_1 \otimes h = h$. □

Notion of a Cyclic Crossed Module

Definition

We define a **cyclic crossed module** $\mathbb{X} = (G, H, t, \alpha)$ to be when G and H are cyclic groups, and α is trivial.

Theorem

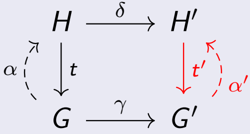
A cyclic crossed module induces an abelian 2-group.

Proof.

We want to show that $(h, g) \otimes (h', g') = (h', g') \otimes (h, g)$.
$(h, g) \otimes (h', g') = (h\alpha_g(h'), gg') = (hh', gg') = (h'h, g'g) = (h'\alpha_{g'}(h), g'g) = (h', g') \otimes (h, g)$. □

Theorem

Given a crossed module $\mathbb{X} = (G, H, t, \alpha)$ and isomorphisms $\delta : H \to H'$ and $\gamma : G \to G'$, we can induce a crossed module structure as follows:



We define $t' := \gamma \circ t \circ \delta^{-1}$ and $\alpha' := \delta(\alpha_{\gamma^{-1}}(\delta^{-1}))$

Remark: The induced crossed module structure $\mathbb{X}' = (G', H', t', \alpha')$ creates an isomorphism of crossed modules $(\gamma, \delta) : \mathbb{X} \to \mathbb{X}'$

Example: Inducing a Crossed Module Structure

Example

In group theory, every finite abelian group is isomorphic to a direct sum of cyclic groups of prime order.

$$H \cong \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_k} \qquad G \cong \mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \oplus \cdots \oplus \mathbb{Z}_{q_m}$$

If the crossed module $\mathbb{X}$ is an abelian crossed module, we can induce a crossed module structure from such isomorphisms.

$$\begin{array}{ccc} H & \xrightarrow{\delta} & \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \cdots \oplus \mathbb{Z}_{p_k} \\ \downarrow t & & \downarrow t' \\ G & \xrightarrow{\gamma} & \mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \cdots \oplus \mathbb{Z}_{q_m} \end{array}$$

However, is this enough?

Direct Sum of Cyclic Crossed Modules

Definition

We define a **direct sum of cyclic crossed modules** by the following intuition:

$$\begin{array}{c} \mathbb{Z}_{p_1} \\ \downarrow t_1 \\ \mathbb{Z}_{q_1} \end{array} \oplus \begin{array}{c} \mathbb{Z}_{p_1} \\ \downarrow t_2 \\ \mathbb{Z}_{q_2} \end{array} \oplus \cdots \oplus \begin{array}{c} \mathbb{Z}_{p_n} \\ \downarrow t_n \\ \mathbb{Z}_{q_n} \end{array} := \begin{array}{c} \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_n} \\ \downarrow \overline{t} \\ \mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \oplus \cdots \oplus \mathbb{Z}_{q_n} \end{array}$$

We define $\overline{t}$ as $(t_1, t_2, \ldots, t_n)$.

Notice

The group action α is omitted in this definition since it's defined to be the trivial action $\alpha_g(h) = h$.

Example

Consider the direct sum of cyclic crossed modules $(\mathbb{Z}_2, \mathbb{Z}_2, \mathrm{id})$ and $(\langle 0 \rangle, \mathbb{Z}_2, \mathrm{id})$.

$$\begin{array}{c} \mathbb{Z}_2 \\ \downarrow \mathrm{id} \\ \mathbb{Z}_2 \end{array} \quad \oplus \quad \begin{array}{c} \langle 0 \rangle \\ \downarrow \mathrm{id} \\ \mathbb{Z}_2 \end{array} \quad := \quad \begin{array}{c} \mathbb{Z}_2 \oplus \langle 0 \rangle \\ \downarrow \, \bar{t} = (\mathrm{id}, \mathrm{id}) \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{array}$$

The direct sum of crossed modules is isomorphic to $(\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2, t)$, where $t(1) = (1, 0)$.

$$\begin{array}{ccc} \mathbb{Z}_2 \oplus \langle 0 \rangle & \xrightarrow{\cong} & \mathbb{Z}_2 \\ \downarrow \bar{t} & & \downarrow t \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \xrightarrow{\cong} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{array}$$

Isomorphisms to Direct Sums (pt. 2)

Notice

Crossed modules with different homomorphism t may be isomorphic to the same direct sum of cyclic crossed modules.

Consider the previous diagram, but let $t(1) = (1,1)$ for $(\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2, t)$.

$$\begin{array}{ccc} \mathbb{Z}_2 \oplus \langle 0 \rangle & \xrightarrow{\cong} & \mathbb{Z}_2 \\ \downarrow{\scriptstyle \overline{t}=\mathrm{id}} & & \downarrow{\scriptstyle t} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \xrightarrow{\cong} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{array}$$

Isomorphism at the bottom is characterized by $(1,0) \mapsto (1,1), (1,1) \mapsto (1,0)$.

Finding the "right" isomorphisms, given some abelian crossed module, is a lot more challenging.

Open Research Questions

- We can induce a crossed module structure via isomorphism by the Fundamental Theorem of Finite Abelian Groups. Is that induced structure isomorphic to a direct sum of cyclic crossed modules (if so, is there an algorithm)?
- What is the intuition for a 2-group having **equivalences** in the crossed module setting? Are there analogues for these theorems for equivalences?
- A 2-group need not have strict inverses for objects (i.e. for a **weak inverse** g^{-1} to g, $g \otimes g^{-1} \cong 1 \cong g^{-1} \otimes g$). What further theorems can be obtained from this structure?

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Thank you!