Finite 2-Groups

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Introduction

In abstract algebra, a **group** is an algebraic structure that can represent the abstraction of symmetries of a given set of objects. To represent the abstraction of symmetries on both objects and the relations (or morphisms) between objects, a **2-group** is constructed. The project provides insight into analogues of known theorems under group theory in the strict 2-group setting, such as the fundamental theorem of finite abelian groups and the isomorphism theorems. To find such analogues, a **crossed module** is constructed, which represents the group structure of objects and morphisms for a 2-group. In that setting, we are able to construct notions of "crossed submodules" to represent sub-"2-groups", which helps arise concepts of kernel, image, and normal sub-2-groups.

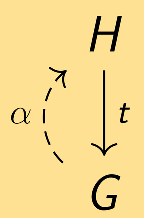
List of Definitions

Definition (2-Group)

A (strict) **2-group** is a category $(\mathcal{G}, \otimes)$, with $\otimes$ denoting the unital and associative binary operation between two objects or two morphisms, for which objects and morphisms are invertible.

Definition (Crossed Module)

A **crossed module** $\mathbb{X} = (G, H, t, \alpha)$ refers to groups G and H, a homomorphism $t : H \to G$, and a group action $\alpha : G \to \mathrm{Aut}(H)$:



In addition, t, α satisfies *equivariance* $(t(\alpha_g(h)) = gt(h)g^{-1})$ and the *Peiffer identity* $(\alpha_{t(h)}(h') = hh'h^{-1})$ for $g \in G, h, h' \in H$.

Definition (Morphism of Crossed Modules)

A **morphism of crossed modules** $f : \mathbb{X} \to \mathbb{X}'$ refers to a pair of homomorphisms γ, δ where $\gamma : G \to G'$ and $\delta : H \to H'$ such that $\gamma \circ t = t' \circ \delta$ and $\delta(\alpha_g(h)) = \alpha'_{\gamma(g)}(\delta(h))$.

Definition (Crossed Submodules, Normal Crossed Submodules)

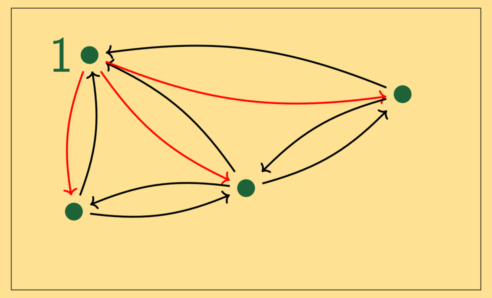
A **crossed submodule** $\mathbb{X}^* = (G^*, H^*, t^*, \alpha^*)$ of a crossed module $\mathbb{X}$ is formed when G^* is a subgroup of G, H^* is a subgroup of H, t^* is the restriction of t on H^*, and α^* is restriction of α on G^*, H^*.

The crossed submodule is **normal** if G^* is a normal subgroup, $\alpha_g(h^*) \in H^*$, and $\alpha_{g^*}(h)h^{-1} \in H^*$.

Relationship of 2-Groups and Crossed Modules

2-Groups Induce Crossed Modules

Suppose a 2-Group $(\mathcal{G}, \otimes)$ is given below.

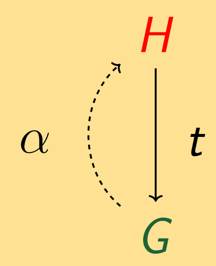


$(\mathcal{G}, \otimes)$

To construct a crossed module $\mathbb{X} = (G, H, t, \alpha)$, let the objects be elements in G, and let the unit-source morphisms be elements in H. Let $t : H \to G$ be the targets of the unit-source morphisms, and let α be defined by conjugation $\mathrm{id}_g \otimes h \otimes \mathrm{id}_{g^{-1}}$.

Crossed Modules Induce 2-Groups

Suppose a crossed module $\mathbb{X} = (G, H, t, \alpha)$ is given.



To construct a 2-Group $(\mathcal{G}, \otimes)$, converse argument holds for constructing objects and unit-source morphisms in $\mathcal{G}$ from G, H, t. Let $\otimes$ be defined by the semi-direct product where $(h, g) \otimes (h', g') = (h\alpha_g(h'), gg')$.

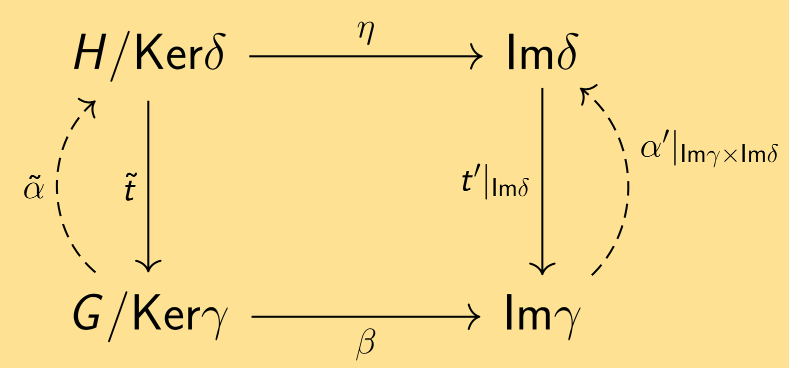
First Isomorphism Theorem for Crossed Modules

Lemma 1.1: Given a morphism of crossed modules $f : \mathbb{X} \to \mathbb{X}'$, a **kernel crossed submodule** $\mathrm{Ker}f = (\mathrm{Ker}\gamma, \mathrm{Ker}\delta, t, \alpha)$ can be defined. Furthermore, the kernel crossed submodule is normal.

Lemma 1.2: From the morphism of crossed modules, an **image crossed submodule**, denoted as $\mathrm{Im}f = (\mathrm{Im}\gamma, \mathrm{Im}\delta, t'|_{\mathrm{Im}\delta}, \alpha'|_{\mathrm{Im}\gamma \times \mathrm{Im}\delta})$, exists.

Lemma 2: Given a crossed module $\mathbb{X} = (G, H, t, \alpha)$ and a normal crossed submodule $\mathbb{X}^*$, we can construct a **quotient crossed module** $\mathbb{X}/\mathbb{X}^* = (G/G^*, H/H^*, \tilde{t}, \tilde{\alpha})$, where $\tilde{t}(hH^*) = t(h)H^*$ and $\tilde{\alpha}_{gG^*}(hH^*) = \alpha_g(h)H^*$.

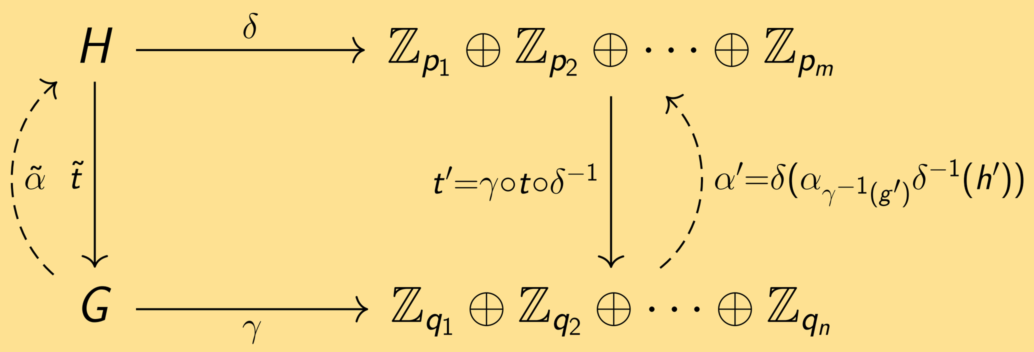
Theorem: Suppose $f = (\gamma, \delta) : \mathbb{X} \to \mathbb{X}'$ is a morphism of crossed modules. Then there exists an isomorphism of crossed modules $(\beta, \eta) : \mathbb{X}/\mathrm{Ker}f \to \mathrm{Im}f$. In particular, the maps are given by $\eta(h\mathrm{Ker}\delta) = \delta(h)$ and $\beta(g\mathrm{Ker}\gamma) = \gamma(g)$.



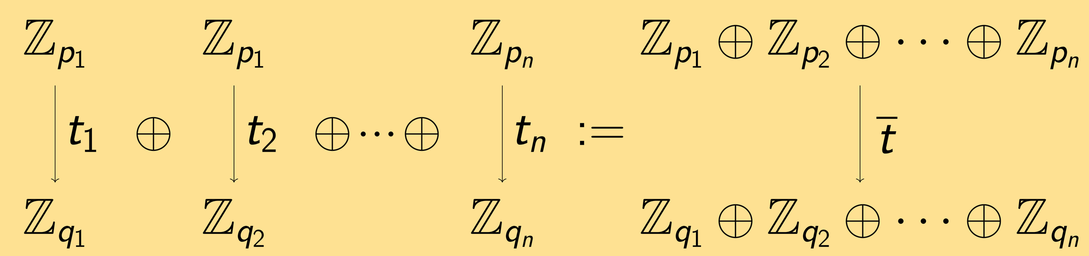
Progress with the Fundamental Theorem of Finite Abelian 2-Groups

A crossed module $\mathbb{X} = (G, H, t, \alpha)$ is **abelian** if G and H are abelian (i.e. elements have the commutative property) and if α is trivial (such that $\alpha_g(h) = h$).

In group theory, every finite abelian group can be written as a direct sum of cyclic groups. We can induce a crossed module structure that consists of such direct sums of cyclic groups as below:



Our notion of a **cyclic crossed module** (G, H, t, α), naively, is if G is generated by some $g \in G$ and H is generated by some $h \in H$. The direct sum of cyclic crossed modules is expressed by the following:

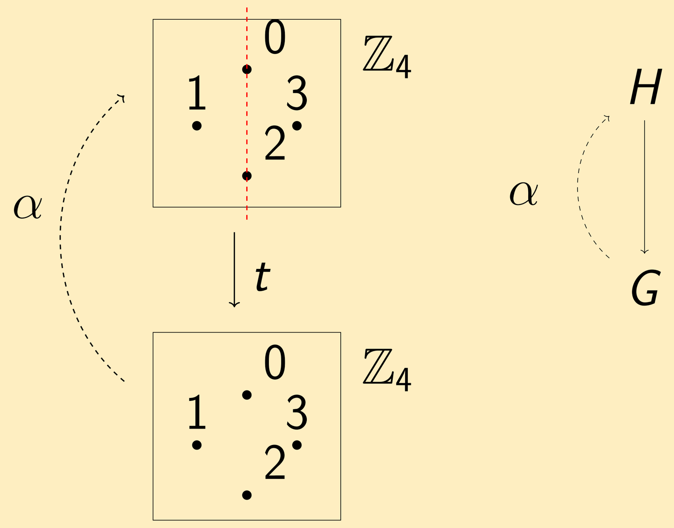


We let $\bar{t} := (t_1, t_2, \ldots, t_n)$.

The question of whether abelian crossed modules have an isomorphism to a direct sum of cyclic crossed modules is still open.

The Smallest Non-Trivial Crossed Module

Consider $\mathbb{X} = (\mathbb{Z}_4, \mathbb{Z}_4, t, \alpha)$, letting $t(1) = 2$ and $\alpha_1(1) = \alpha_3(1) = 3, \alpha_1(3) = \alpha_3(3) = 1$, and $\alpha_2(h) = h$.



Conclusions/Future Work

Since 2-groups induce crossed modules and crossed modules induce 2-groups, we have verified notions of a homomorphism of 2-groups by working in the crossed module setting, and likewise for images, sub-2-groups and normal sub-2-groups, kernels, images, and quotients. Using those definitions, we are able to construct a first isomorphism theorem for crossed modules.

The research questions that are still open are as follows:

- Is our notion of a cyclic crossed module correct, and can that definition be used to construct a crossed module analogue to the fundamental theorem of finite abelian groups?
- Is there a crossed module analogue to the second isomorphism theorem? What is a notion of a "product crossed module" and the intersection of crossed modules?

Nicholas Lear will continue the project using more general definitions of 2-groups, such that inverses of objects in $(\mathcal{G}, \otimes)$ need not be strict (i.e. g' is a weak inverse of g if $g \otimes g'$ and $g' \otimes g'$ are isomorphic to the unit object 1).

Acknowledgments

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References

- John C. Baez and Aaron D. Lauda. Higher-dimensional Algebra V: 2-Groups, 2004.
- M. Ladra and A. R.-Grandjean. Crossed modules and homology. *J. Pure Appl. Algebra, 95(1):41–55*, 1994.
- Behrang Noohi. Notes on 2-groupoids, 2-groups and crossed-modules, 2008.
- Sven-S. Porst. Strict 2-groups are crossed modules, 2008.