Principal Minors of the Fourier Matrix

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Introduction

The aim of this project is to explore a conjecture (Main Conjecture below) involving principal minors of the Fourier matrix (also called the DFT matrix). The conjecture arose in the context of some recent research and some informal numerical work has been done on this conjecture and it has been confirmed up to $N = 20$. We started with a theorem found in [1] and [2].

Key Definitions

Definition (N^{th} Root of Unity)

For $N \in \mathbb{N}$, the *primitive N-th root of unity* is defined by

$$\omega_N = \omega = e^{2\pi i/N} = \cos(2\pi/N) + i\,\sin(2\pi/N),\ \ i^2 = -1.$$

Definition (The Discrete Fourier Transform Matrix)

For $N \in \mathbb{N}$, the $N \times N$ *Fourier (or DFT) matrix*, W_N, is defined by

$$W_N = \left(\omega_N^{jk}\right)_{j,k=0}^{N-1}$$

When N is understood, we write $W_N = W$ and $\omega_N = \omega$.

Definition (Principal Submatrices and Minors)

Given $I \subseteq \{0, 1, \ldots, N-1\}$, and M an $N \times N$ matrix, M^I is the $|I| \times |I|$ submatrix of M whose column and row indices both come from I. The *principal minor* of M corresponding to I is $\det(M^I)$.

Main Conjecture

For every $N \in \mathbb{N}$, there exists a permutation σ of $\{0, 1, \ldots, N-1\}$ such that every principal minor of the matrix W_N^σ is nonzero. Here W_N^σ is the DFT matrix W_N whose rows have been permuted by σ.

Some known results

Theorem

(Chebotarev) If N is prime, then every minor of W_N is nonzero. This includes all principal minors.

Theorem

(Tao, Evans and Isaacs) Given $I = \{a_1, a_2, \ldots, a_n\} \subseteq \{0, 1, \ldots, N-1\}$, if

$$\Phi_N(1) \nmid \frac{\prod_{i<j}(a_j - a_1)}{(n-1)!(n-2)!\cdots(2)!(1)!}$$

then for any σ, $\det((W_N^\sigma)^I) \neq 0$. Here $\Phi_N(x)$ is the N^{th} cyclotomic polynomial.

- If $N = p$ prime, then $\Phi_N(1) = p$ thereby proving Chebotarev's Theorem.
- If $N = p^k$, then also $\Phi_N(1) = p$.

Results

Most of our work has been to prove some theorems to be able to work with principal submatrices and minors and to numerically confirm the conjecture past $N = 20$. We provide the theorems and proof outlines below.

Theorem 1: The Twin Singularity Theorem
Theorem 2: The Sliding Theorem
Theorem 3: Principal Vandermonde Submatrices

Theorems and Proof Outlines

Theorem

(Twin Singularity Theorem) Given $I \subseteq \{0, 1, \ldots, N-1\}$, and any σ, let $A = (W_N^\sigma)^I$ and $B = (W_N^\sigma)^{I^c}$. Then $\det(A)$ and $\det(B)$ are either both zero or both nonzero.

Outline:

1. The vector $v \in \ker(A)$ projected into $v \mapsto x \in \mathbb{C}^n$ where x has non-zero entries associated to the column vectors $\{a_i\}$. We know that v exist since A is singular and now we want to use the fact that W_N is invertiable.
2. We know that $W_N x = \hat{x}$ is not zero since W_n is invertible and since $v \in \ker(A)$, there are 0's in all of the coordinates of $\hat{x}$ that correspond to the column vectors $\{a_i\}$.
3. The next step is to multiply both sides by $W_N^{-1} = \overline{W_N}$ and take the conjugate. The key is that conjugation keeps all of the non-zero entries in their place and will not send them to zero.
4. We then project our conjugated vector down and show that the principle submatrix associated to the column vectors $\{a_i\}^c$ has a non-trivial kernel, which means that it is singular.

Theorem

(Sliding Theorem) Let W_N be the $N \times N$ DFT matrix and $I = \{a_0, \ldots, a_r\} \subseteq \{0, \ldots, N-1\} \subset \mathbb{N}$ be and ordered set of indices such that $a_0 \neq 0$ and let $I_0 = \{0, a_1 - a_0, \ldots, a_r - a_0\}$. The principal submatrix, W_N^I is non-singular if and only if $W_N^{I_0}$ is non-singular.

We did this by factoring out ω's from the row and columns and have an explicit formula for relating the two principle minors. Here is the formula:

$$\det(W_N^I) = \prod_{i=0}^{r}\omega^{a_i}\prod_{i=1}^{r}\omega^{a_i - a_0}\det(W_N^{I_0})$$

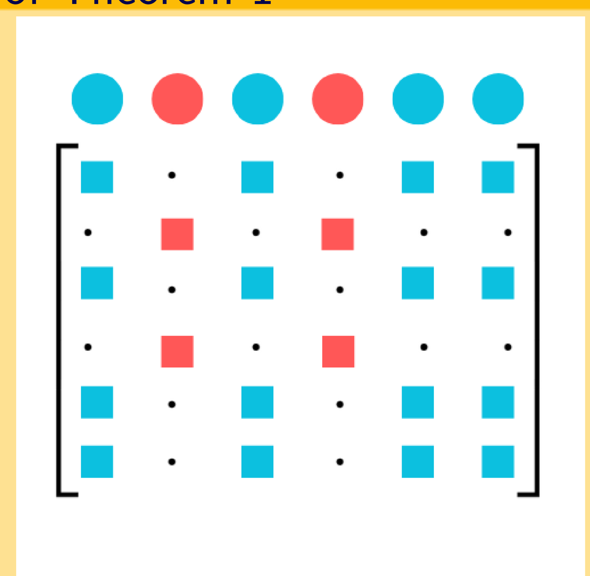
Theorem

(Principal Vandermonde Submatrices)Let W_N be the $N \times N$ DFT matrix and let $G = \langle\omega\rangle$ and let $k = |H|$ where H is a subgroup of G. The principal submatrix, W_N^I, associated to the indices, $I = \{0, N/k, \ldots, (k-1)N/k\}$ is Vandermonde.

For this we used Lagrange's theorem and saw that this selection of columns makes a Vandermonde matrix. The proof follows from the selection of column indices and the fact that all of the subgroups will be cyclic.

Graphics

Visualization of Theorem 1



Visualizaition of Roots of Unity

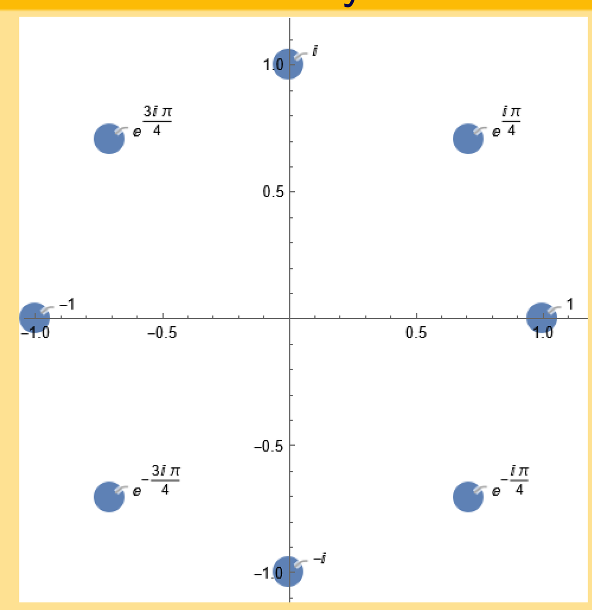


Figure: Roots of Unity for $N = 8$

Numerical Findings

- We have verified the conjecture up to $N = 30$.
- We have also computed the total number of valid permutations for up to N equals 12: $\{(4:16), (6:144), (8:2304), (9, 46656), (10, 43400), (12, 38880)\}$. Note that if N is prime then Chebotarev's theorem says the number of valid permutations is $N!$.
- We observe that if $N = pq$, p, q distinct primes then the identity permutation appears to work.

Conclusions/Future Work

We seek to prove the following conjectures.

Conjecture 2. if $N = pq$, p, q distinct primes then every principal minor of W_N is nonzero. This conjecture is informed by numerical evidence.

Conjecture 3. If $N = p^k$, p prime then there is an explicit permutation σ that satisfies the Main Conjecture. Here we hope to leverage the Tao/Evans-Isaacs formula.

Future work will include checking the Main Conjecture for larger integers and explore the above conjectures. We have begun to build the techniques to prove/disprove this conjecture and to find the limits of the current techniques [1][2].

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References

[1] T. Tao, "An uncertainty principle for cyclic groups of prime order," Math. Res. Lett. 12 (2005), no. 1, 121–127.

[2] R. J. Evans and I. M. Isaacs, "Generalized vandermonde determinants and roots of unity of prime order," Proc. Amer. Math. Soc.58(1976), 51–54.