The Kelly Criterion

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Introduction

In 1956, J. L. Kelly Jr. described what is now known as The Kelly Criterion. This Criterion is a formula that determines the percentage of one's wealth that should be used in order to maximize the utility of long-term growth (utility is the log of wealth). The Kelly Criterion was originally applied to casino gambling, and later to the stock market. In our research, we applied the Criterion to the stock market and to insurance (we may be the first to have done the latter). For the stock market, we used the Geometric Brownian Motion (GBM) Model. This model was used in the Nobel-Award winning Black-Scholes stock option pricing analysis. We also applied the Kelly Criterion to insurance. Here Jensen's Inequality is often cited as a justification for purchasing insurance. We build on that by addressing the question of how much insurance should the consumer purchase. Our goal with this project was to provide empirical evidence that the Kelly Criterion is an acceptable optimal growth strategy in various real-world risk-reward situations.

Applicable Definitions

Definition (Kelly Criterion)

We can understand the Kelly Criterion as a function that finds a K that maximizes the Geometric Growth of Wealth.

$$Maximize\{E[G(K)], K\}$$

- K is the percentage of wealth that one can use
- $G(K)$ measures the Geometric Growth Rate.

The geometric growth rate is the percentage increase in the log of wealth.

Definition (Geometric Growth Rate)

The Geometric Growth Rate is the ratio between the Weighted Geometric Mean of all possible wealth after a bet and the wealth before the bet.

$$E[G(K)] = e^{-\ln(w)+\Sigma_{i=1}^{n} p_i \ln(a_i)}$$

- $G(K)$ measures the Geometric Growth Rate.
- w is the principal or initial wealth
- n is the number of possible outcomes
- a_n is the nth possible outcome of a bet
- p_n is the probability of the nth possible outcome occurring

Definition (Logarithmic Utility)

Logarithmic Utility uses the Natural Logarithm to quantify how useful some goods or services are to an individual. We calculate utility as the natural log of wealth.

Stock Market Model

The GBM Model measures in continuous time and assumes that the proportional return of a stock is log-normally distributed. The following equation reflects these aspects:

$$\frac{S_T}{S_o} = \mu T + \sigma \epsilon \sqrt{T}$$

- S_n is price of the stock at the nth time interval
- T is number of time intervals after the initial price
- μ is expected rate of return after one interval
- σ is standard deviation of return rates after one interval
- ϵ is a randomly drawn value from the standard normal distribution

Since there are risk-free growth options in the real world, we included them in our model. In fact, however much wealth we withheld from Stock Investments, we put into this risk-free option. The growth rate of this option is represented by r.

Insurance Model

Our Insurance Model is a Stochastic Process that represents the client's wealth as it changes from month to month. We did this because, in Fire Insurance, the change in the client's wealth could be different if a fire occurs. Since the insurer must compensate for fire damages, which could vary greatly, we simplified the model by assuming that the cost of any fire damage is equal to the house's value. The following expressions show what the client's wealth would be if a fire occurs and if it doesn't:

$$1)\, W_{1+t} = W_t - hrK - h + hK \qquad 2)\, W_{1+t} = W_t - hrK$$

- t is the number of Months after the initial price
- W_n is the client's wealth after the nth Month
- h is the value of the client's house
- r is the rate cost of insurance per dollar
- K is the coverage that the client bought

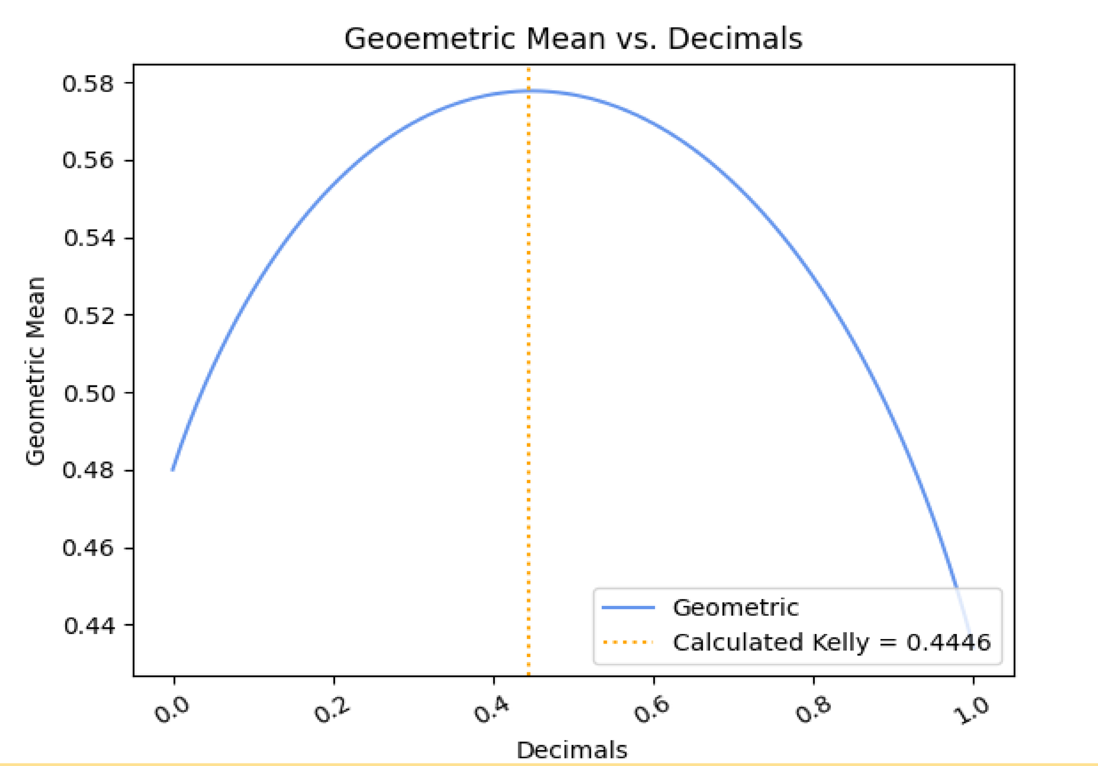
Kelly Application of the Stock Market

Since Stock Investments have an infinite number of outcomes, we must simplify the problem. We accomplished this by creating a "winning" and "losing" scenario, which correspond to ϵ being 1 or -1. Furthermore, we restrict ϵ to only be one of these values. This allows us to easily calculate the Geometric Growth Rate. After manipulating the GBM Equation, we find the following formula for the Logarithmic Utility of the Geometric Growth Rate as T approaches infinity.

$$U_\infty(K) = r + K(\mu - r) - \frac{\sigma^2 K^2}{2}$$

By maximizing this equation, we find the Kelly Formula for the Stock Market:

$$K = \frac{\mu - r}{\sigma^2}$$



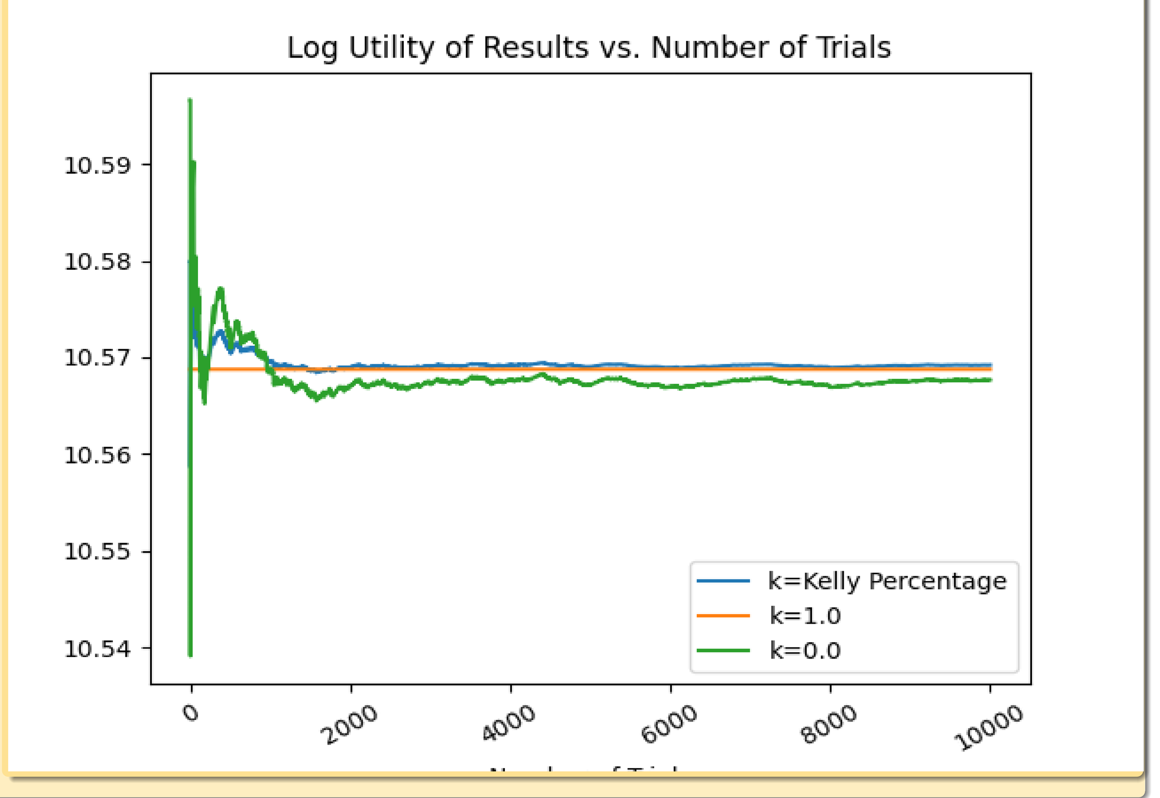
Kelly Application of Fire Insurance

Due to the simplicity of our Insurance Model, we can easily find a formula for the Logarithmic Utility of the Geometric Growth Rate.

$$U(K) = -\ln(w) + p\ln(w - hrK - h + hK) + (1-p)\ln(w - hrK)$$

Due to the Logarithms in this formula, there are values of K where $U(K)$ is undefined. After maximizing this equation, we find:

$$K = \begin{cases} Undefined & 0 \le r \le 1, w \le hr \\ Undefined & r \ge 1, w \le h \\ 0 & r \ge 1, w \ge h \\ 0 & 0 \le r \le 1, h \le w, w \ge \frac{hr(1-p)}{r-p} \\ 1 & 0 \le r \le 1, w \ge hr, r \le p \\ \frac{w(p-r)+hr(1-p)}{hr(1-r)} & Else \end{cases}$$



Definition (Jensen's Inequality)

Jensen's Inequality states that for any random variable X and any concave function ϕ, we can conclude:

$$\phi(E[X]) \ge E[\phi(X)]$$

Since the Kelly Criterion uses the logarithm function to calculate utility, we use it as ϕ. Since ϕ is concave, we can conclude that the expected utility of your wealth after some decision is always less than or equal to the utility of your expected wealth. This grants users an incentive to maximize their expected utility to garner better yields.

Conclusions/Future Work

We used Python-based simulations to test if the percentage of wealth indicated by the Kelly Criterion produces optimal financial growth for investors. Regarding the Stock Market, our simulations confirmed that Thorpe's formula for estimating the Kelly Percentage produces optimal long-term growth. Regarding fire insurance, our simulations indicated that consumers do not need to fully insure their homes. In future research, the Kelly Criterion could be applied to a portfolio consisting of more investment options. If we are correct that consumers should not fully insure their homes, future work could also investigate the savings to the economy if banks did not require full insurance.

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