Universal Partial Cycles in Higher Dimensions

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Friday, 1st December, 2023

Introduction

This project studies the combination of two generalizations of the de Bruijn Cycle: the de Bruijn torus and the universal partial cycle. We show the existence of novel objects – universal partial tori – and describe some of their properties.

Definition (Universal Partial Torus)

A *universal partial torus* – shortened to "uptorus" – of window size $w \times \ell$ over alphabet $\mathcal{A} \cup \{\diamond\}$ is an array where both the rows and columns of the array are considered cyclically and in which every rectangular array of size $w \times \ell$ over alphabet $\mathcal{A}$ appears exactly once, treating $\diamond$ as matching any symbol in $\mathcal{A}$. (c.f. (3) and (2))

Definition (Universal Partial Family)

A *universal partial family* for $\mathcal{A}^\ell$ is a set $\mathcal{S}$ of cyclic strings on $\mathcal{A} \cup \{\diamond\}$ such that every element of $\mathcal{S}$ has the same length, and every word in $\mathcal{A}^\ell$ is covered by exactly one element of $\mathcal{S}$, and is only covered once in that element.

Definition (Alternating Word)

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite alphabets. An alternating word on the alphabet pair $(\mathcal{A}, \mathcal{B})$ is a string of length $2n+1$ which alternates between elements of $\mathcal{A}$ and elements of $\mathcal{B}$ (beginning and ending with an element of $\mathcal{A}$). (3)

Definition (Alternating de Bruijn Sequence)

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite alphabets. An alternating de Bruijn (DB) sequence is a cyclic string which alternates between elements of $\mathcal{A}$ and elements of $\mathcal{B}$ and contains every alternating word on $(\mathcal{A}, \mathcal{B})$ of length $2n+1$ exactly once as a substring. (3)

Minimal Example of an Uptorus

Given $\mathcal{A} = \{0, 1\}$ and window size $w \times \ell = 2 \times 2$, there are universal partial tori with exactly one wildcard each:

$$\begin{matrix} \diamond & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix} \qquad \begin{matrix} \diamond & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{matrix} \qquad \begin{matrix} \diamond & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} \diamond & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}$$

We discovered these universal partial tori by an exhaustive computer search of the small solution space.

They are minimal in that no uptorus can have fewer wildcards (as it would then be a de Bruijn torus), no nontrivial uptorus can cover a smaller window (as it would then be a universal partial cycle), and no nontrivial uptorus can have smaller dimensions (as it would either repeat windows or omit windows).

Constraints on the Size of Uptori

A de Bruijn torus with window size $w \times \ell$ over alphabet $\mathcal{A}$ is an array with exactly $|\mathcal{A}|^{w\ell}$ cells. Because de Bruijn tori are rectangular, the factorization of $|\mathcal{A}|^{w\ell}$ provides possibilities for the dimension of the torus.

Theorem (Minimal Uptorus Dimensions)

An uptorus for $r \times c$ submatrices has dimensions at least $(r+1) \times (c+1)$.

Theorem (Wildcards and Uptorus Dimensions)

A de Bruijn torus with windows size $w \times \ell$ over alphabet $\mathcal{A}$ has $|\mathcal{A}|^{w\ell}$ cells. An uptorus with the same window size and alphabet has fewer cells. A window containing d wildcards covers $|\mathcal{A}|^d$ distinct submatrices, and consequently each non-overlapping wildcard reduces the size of the potential uptorus by $|\mathcal{A}|^{w\ell}$ cells.

Universal Partial Families

In (1), the authors identify an upcycle over the alphabet $\{0, 1, 2, 3\}$ for windows of length four:

001⋄110⋄003⋄112⋄021⋄130⋄023⋄132⋄201⋄310⋄203⋄312⋄221⋄330⋄223⋄332⋄

We conjectured that we could produce a universal partial family by slicing that into equally sized cyclic strings and verified our conjecture by computer for cyclic strings of length 8, 16, and 32 (the slicing works whatever index is chosen for the first cut). For example, the four member family

001⋄110⋄003⋄112⋄
021⋄130⋄023⋄132⋄
201⋄310⋄203⋄312⋄
221⋄330⋄223⋄332⋄

is a universal partial family. The same slicing construction failed on the binary upcycle for words of length 8:

0000010⋄1111101⋄0010010⋄1101101⋄1110000⋄0001111⋄1110011⋄0101100⋄
1110010⋄0101001⋄1000110⋄0100001⋄1011110⋄0101101⋄0000110⋄1101001⋄

so we conjecture that the slicing construction only produces universal partial families if the original upcycle is the product of the alphabet multiplier theorem.

Analogue of the Construction by Kreitzer et al.

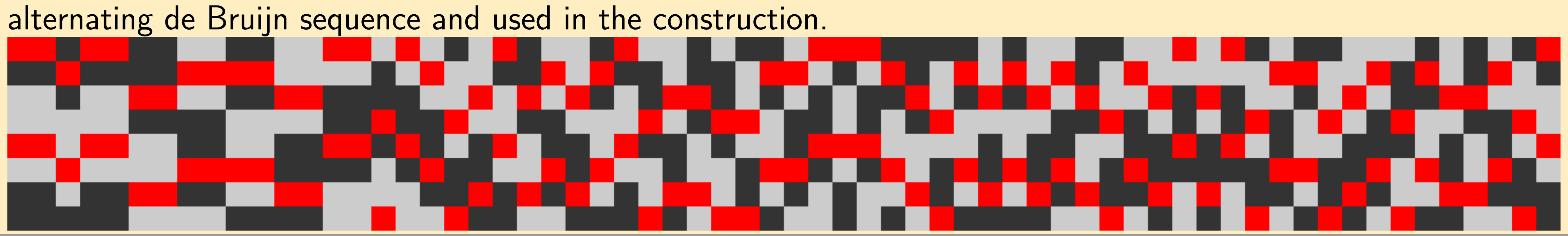
Kreitzer et al. (3) constructed de Bruijn tori using alternating de Bruijn sequences and de Bruijn families. A de Bruijn family is like a universal partial family, but it has no wildcards. The alphabet pair that the alternating DB sequence uses is $(\mathcal{S}, \Pi)$, where $\mathcal{S}$ is the de Bruijn family and Π is the set of rotations on each element of $\mathcal{S}$. The construction involves cyclically shifting elements of $\mathcal{S}$ according to the alternating de Bruijn sequence and then stacking them.

We found that the method works when $\mathcal{S}$ is replaced with a universal partial family. The result includes wildcards and should be an uptorus. An upcycle can be viewed as a universal partial family with a single element.

We have conjectured conditions that should guarantee that the process produces an uptorus and computationally verified a few examples. In the future we will try to prove that the method works when the conditions are met.

Example: An 8×64 Uptorus Covering 4×3 Windows over the Binary Alphabet

Black pixels represent 0s, gray pixels 1s, and red pixels ⋄s.
The upcycle 001⋄110⋄ (mentioned in (1; 2)) was used. A de Bruijn cycle generated at (4) was converted to an alternating de Bruijn sequence and used in the construction.



Conclusions/Future Work

We have many open questions that we hope to answer:

1. Is there an uptorus for the 3×2 window over a binary alphabet? If not, why not?
2. When does the uptorus version of Kreitzer et al.'s method work? For what alphabet and submatrix sizes?
3. For what alphabet and word sizes is it possible to partition an upcycle into a universal partial family?
4. Are there universal partial "hypertori" that occupy 3 or more dimensions? If so, is there an algorithm to generate them?

We also have several conjectures for which we are composing proofs:

1. There are infinitely many uptori not discoverable by the method in (3) that uses alternating de Bruijn sequences to construct de Bruijn tori.
2. There is some method of constructing an uptorus that begins with a regular de Bruijn torus and inserts wildcards reducing the dimensions.

Acknowledgments

Many thanks to Matthew Kearney and Dr. Kirsch for their excellent questions, suggestions, conjectures, and advice.

References

[1] Dylan Fillmore, Bennet Goeckner, Rachel Kirsch, Kirin Martin, and Daniel McGinnis. The existence and structure of universal partial cycles, 2023. arXiv:2310.13067.

[2] Bennet Goeckner, Corbin Groothuis, Cyrus Hettle, Brian Kell, Pamela Kirkpatrick, Rachel Kirsch, and Ryan Solava. Universal partial words over non-binary alphabets. *Theoretical Computer Science*, 713:56–65, 2018. doi:10.1016/j.tcs.2017.12.022.

[3] Matthew Kreitzer, Mihai Nica, and Rajesh Pereira. Using alternating de Bruijn sequences to construct de Bruijn tori, 2023. arXiv:2306.01498.

[4] Joe Sawada. De Bruijn sequence and universal cycle constructions, 2023. URL: http://debruijnsequence.org.