Optimal Embedding of a Homogeneous Tree in the Hyperbolic Disk

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Introduction

A homogeneous tree of even degree $s = 2t$ is a Cayley graph of a free group on t generators. In [1], Cohen and Colonna gave conditions under which a homogeneous tree of even degree could be embedded in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

in such a way that the automorphisms induced by rotations and by translations by group elements may be represented by disk automorphisms in such a way that each edge in the embedded tree has the same hyperbolic length. In the optimal case, in which the set of limit points of the vertices is the entire unit circle, the bounded analytic functions on the disk are determined by their values on the vertices of the embedded tree.
We provided a visualization of the embedding for different degrees of homogeneity.

Definition (Homogeneous Tree)

A tree is a connected locally finite graph with no loops, which as a set, we identify with the set of its vertices.
A homogeneous tree is a tree whose vertices have the same number d of neighbors.
The number d is called the degree of the tree.

Definition (Hyperbolic Metric and Geodesics)

The mapping $\rho : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ defined by

$$\rho(z, w) = \frac{1}{2} \ln \left[\frac{1 + \left|\frac{z-w}{1-\bar{z}w}\right|}{1 - \left|\frac{z-w}{1-\bar{z}w}\right|} \right] \quad \text{for} \quad z, w \in \mathbb{D},$$

is a distance on $\mathbb{D}$ called the hyperbolic or Poincaré metric.
In particular,

$$\rho(z, 0) = \frac{1}{2} \ln \left[\frac{1 + |z|}{1 - |z|} \right] \quad \text{for} \quad z \in \mathbb{D}.$$

The geodesics (i.e. the lines of minimal length) are either circular arcs orthogonal to the unit circle $\partial\mathbb{D}$ or diameters if they contain the origin.

Definition (Embedding)

An embedding of a homogeneous tree T into the disk $\mathbb{D}$ is a injective function $\Phi : T \to \mathbb{D}$ that maps every edge of T onto a geodesic arc having a suitable prescribed hyperbolic length.

Definition (Optimal Embedding)

An embedding is said to be optimal if the set of limit points of the vertices in the embedded tree is the entire unit circle.

Free Group Representation of the Tree in the Even Degree Case

View T as a free group on generators, $x_1, x_2, \ldots, x_t$, where $d = 2t$. The vertices correspond to the root e identified with the origin. The neighbors of e are $x_1, \ldots, x_t, x_1^{-1}, \ldots, x_t^{-1}$. The next set of neighbors are obtained by applying the group action: x_1^2, $x_1x_2, \ldots, x_1x_t$, $x_1x_2^{-1}, \ldots, x_1x_t^{-1}$, etc.

The group $Aut(\mathbb{D})$ of disk automorphisms of $\mathbb{D}$ is the collection of bijective analytic self-maps of $\mathbb{D}$. They are special types of Möbius transformations S that can be described as

$$S(z) = \lambda \frac{a - z}{1 - \bar{a}z} \quad z \in \mathbb{D}, \text{ where } |\lambda| = 1 \text{ and } |a| < 1.$$

Main tool for the construction

For $0 < r < 1$ and $\zeta = e^{\pi i/t}$, the disk automorphism formula that yields the vertices of T is

$$X_n(z) = \frac{z + r\zeta^{n-1}}{r\bar{\zeta}^{\,n-1}z + 1} \quad z \in \mathbb{D}.$$

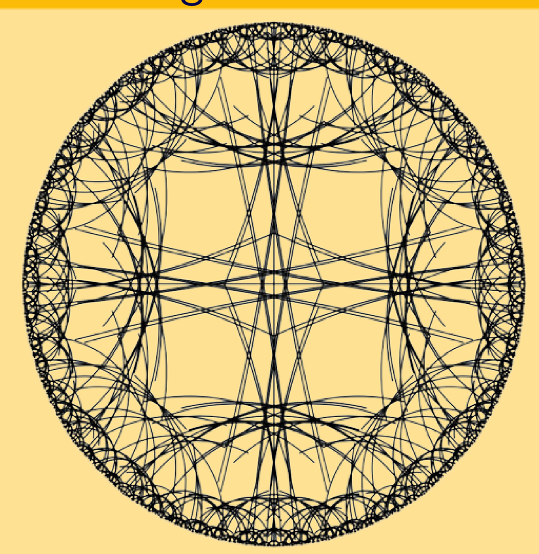
Main Theorem

1. If $0 < r < \cos\frac{\pi}{2t}$, the mapping Φ is not an embedding.
2. If $\cos\frac{\pi}{2t} < r < 1$, Φ is an embedding whose set of limit points does not cover the unit circle.
3. If $r = \cos\frac{\pi}{2t}$, Φ is an optimal embedding.

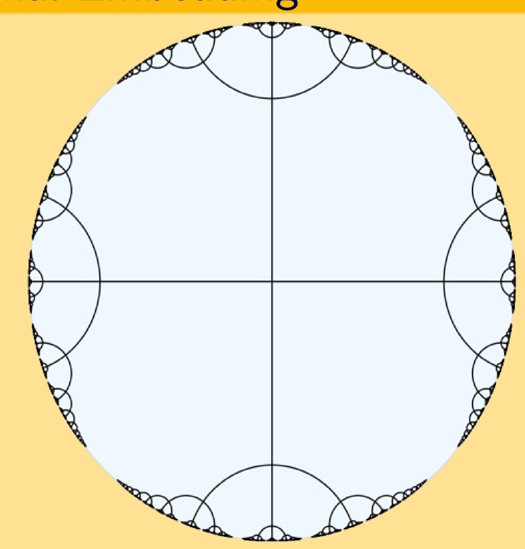
Implementation

To program an algorithm that created a visual representation of the hyperbolic tree, we represented the disk automorphisms as matrices in projective space, otherwise known as CP^2. Projective space can be thought of as a 2D plane, where each point in the plane represents a vector that converges to a point at infinity (i.e. a vanishing point in artist's terminology). We used a recursive algorithm that implemented a left action for each generator for each vertex via matrix multiplication. From here, we are able to find the point in $\mathbb{R}^2$ for each vertex by multiplying the resulting matrix by the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We used the "Geometry Tools" package to connect each vertex with hyperbolic geodesics.

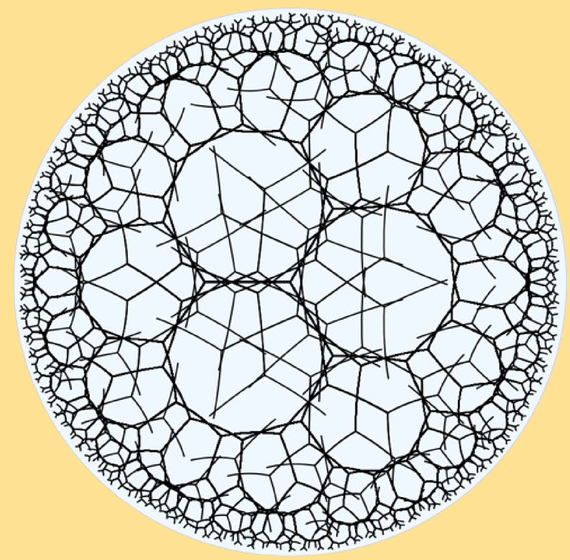
Degree 4 Nonembedding



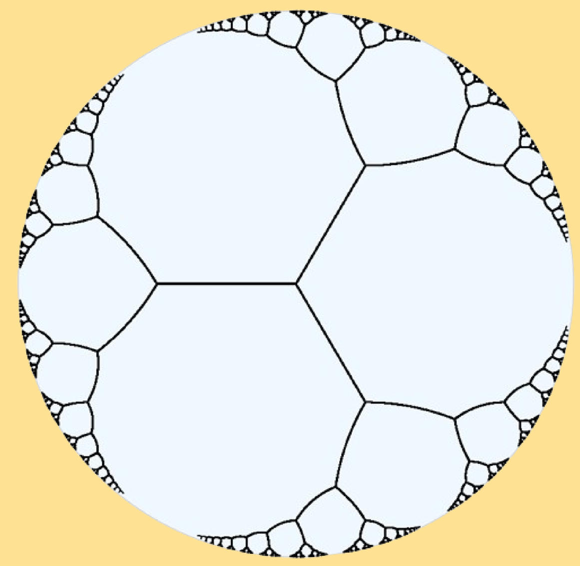
Degree 4 Optimal Embedding



Degree 3 Nonembedding



Degree 3 Optimal Embedding



Conclusions/Future Work

Through the use of pre-existing packages in python, we were able to create visualizations of the optimal embedding of even-degree homogeneous trees in the hyperbolic disk, which was discussed in Dr. Colonna's paper *Embeddings of Trees in the Hyperbolic Disk*. Using our established code, we were then able to extend this to the odd degree case. Future work will include verifying that the embedding is, in fact, optimal, as well as creating visualizations in the upper half plane model.
Further, we plan to determine if we can find an optimal embedding for trees of alternating degree, called "semi-homogeneous".

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References

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