The Kelly Criterion

Dr. Eckley, Dylan S., Nathaniel M., Sarah W.

George Mason University, MEGL

November 30, 2023

Table of Contents

Background Information

Background Information

In 1956, J. L. Kelly Jr. described what is now known as The Kelly Criterion, a formula that is used to experience optimal long-term financial growth. The Criterion was originally applied to casino gambling as a method to determine how much one should bet on any given game. Naturally, the Criterion was later applied to the biggest casino in the world, the Stock Market.

Due to this topic's intimate connection with finance, many economic terms and ideas become vital to research about the Kelly Criterion. One such term is Utility, which is used extensively in the search for Optimal Growth.

About Our Research

Research Goal

Our goal with this project is to provide empirical evidence that the Kelly Criterion provides optimal growth in various real-world risk-reward situations like the Stock Market and Insurance.

Methodology

We formulate a model for the situation we wish to research. We then create Python code that will simulate possible outcomes for the given scenario. From these outcomes, we measure the average utility of the user's wealth growth. This average changes depending on how much of the user's wealth was used; therefore, we plotted this average against the percentage of wealth used.

We theorize that if the percentage of wealth that the Kelly Criterion advises the user to invest produces the largest average utility, then we have demonstrated the Kelly Criterion does provide optimal growth.

Jensen's Inequality

In the past, Jensen's Inequality has been used in finance circles to explain why one should buy Insurance.

Definition (Jensen's Inequality)

Jensen's Inequality states that for any random variable X and any concave function ϕ, we can conclude:

$$\phi(E[X]) \geq E[\phi(X)]$$

Since we use the logarithm function to calculate utility, we use it as ϕ. Since ϕ is concave, we can conclude that the expected utility of your wealth after some decision is always less than or equal to the utility of your expected wealth. This grants users an incentive to maximize their expected utility to garner better yields.

The Definition of the Kelly Criterion

Definition (Kelly Criterion)

The Kelly Criterion is a formula where, if the possible outcomes and their probabilities are known, then it is possible to determine the percentage of one's wealth that should be invested to maximize long-term growth.

To be specific, the Kelly Criterion is a function that finds a K that maximizes the user's Expected Logarithmic Utility of Wealth's Geometric Growth

$$Maximize\{E[U(K)], K\}$$

- K is a real value between 0 and 1 that represents the percentage of wealth that the user will invest
- $E[U(K)]$ is a function that tracks the Logarithmic Utility of Wealth's Geometric Growth Rate as K varies

What is E[U(K)]?

Definition (E[U(K))

The Expected Logarithmic Utility of Wealth's Geometric Growth Rate calculates the potential Growth Rates from each possible outcome, then calculates the logarithmic utility of each outcome, and then takes the weighted average of those utility values.

$$E[U(K)] = \Sigma_{i=1}^{n}(p_i \cdot \ln(\frac{a_i(K)}{w}))$$

- w is the principal or initial wealth
- n is the number of possible outcomes
- $a_n(K)$ is the nth possible outcome of a bet
- p_n is the probability that the nth possible outcome occurs

What is Utility?

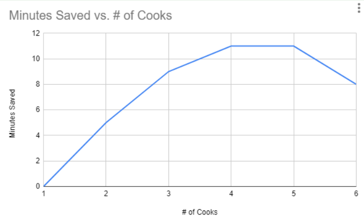
Definition (Utility)

Utility tries to quantify how useful or beneficial some goods and services are to an individual. In our case, we measure the Utility of Wealth.

Since usefulness is very subjective, there is no universal formula for utility; however, a defining feature of utility is the concept of Diminishing Returns.

Definition (Diminishing Returns)

Diminishing Returns refers to the phenomenon where the increase in a quantity of some good or service, results in a decrease in the growth of that quantity's utility. This results in utility following a concave down shape.



The St. Petersburg Game

To better understand the concept of Utility, we studied the St. Petersburg Paradox.

Game Scenario: The St. Petersburg Game

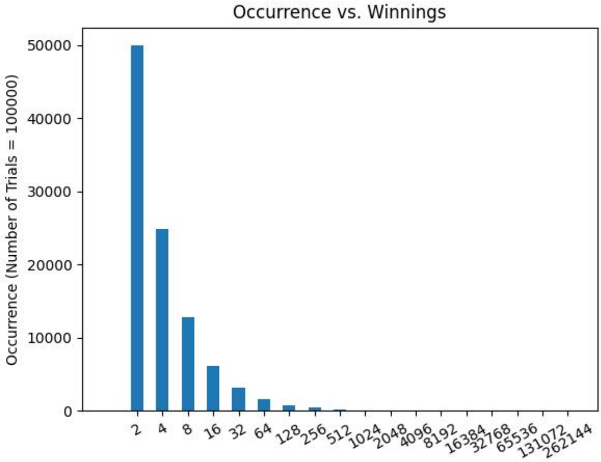
A man approaches a player to play a game! He says that if the player pays him 10 dollars, then he will pay the player 2 dollars, which will double for each consecutive heads he flips on a fair coin.

This means that the man will pay the player 2^{n+1} dollars, where n is the number of consecutive heads flipped. Consequently, this means that the probability of earning 2^{n+1} dollars is $\frac{1}{2^{n+1}}$

Example

After paying the man 10 dollars, he remarkably flipped 5 consecutive heads. The player earned 32 dollars. This means that the player managed to double their money!

Our Results



Occurrence vs. Winnings
Occurrence (Number of Trials = 100000)
50000
40000
30000
20000
10000
0
2
4
8
16
32
64
128
256
512
1024
2048
4096
8192
16384
32768
65536
131072
262144

The St. Petersburg Paradox

Calculating Expected Payout

Let M_n represent the nth possible amount of Money Earned from the game. Let p_n represent the Probability that M_n occurs

Therefore:

$$E[M] = \Sigma_{i=0}^{\infty}(p_i \cdot M_i) = \Sigma_{i=0}^{\infty}(\frac{1}{2^{i+1}} \cdot 2^{i+1})$$
$$E[M] = \Sigma_{i=0}^{\infty}(1) = \infty$$

The Paradox

If we calculate the Expected Payout of the Game, we see that the player should expect to win INFINITE DOLLARS! This means that the player should pay ANY amount to play. This goes against almost every person's intuition, thus the paradox.

Logarithmic Utility Resolves Paradox

In the St. Petersburg game, we have the potential to earn large sums of money, but the usefulness of these sums are questionable.

Definition (Logarithmic Utility)

Logarithmic Utility means that we evaluate the usefulness of some quantity by taking the Natural Log that value.

Calculating Expected Logarithmic Utility of Payout

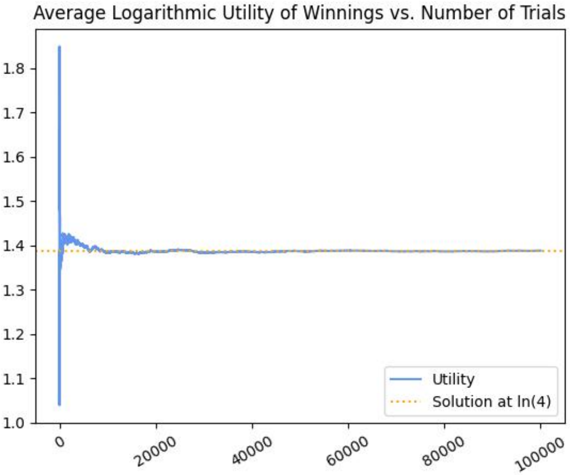
Let U_n represent the Logarithmic Utility of the nth possible amount of Money Earned from the game. Let p_n represent the Probability that U_n occurs

Therefore:

$$E[U] = E[\ln(M)] = \Sigma_{i=0}^{\infty}(p_i \cdot \ln(M_i)) = \Sigma_{i=0}^{\infty}(\frac{1}{2^{i+1}} \cdot \ln(2^{i+1}))$$

$$E[U] = \ln(2) \cdot \Sigma_{i=0}^{\infty}(\frac{i+1}{2^{i+1}}) = 2 \cdot \ln(2) = \ln(4)$$

Our Results



Average Logarithmic Utility of Winnings vs. Number of Trials
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0
20000
40000
60000
80000
100000
Utility
Solution at ln(4)

Stock Market: Geometric Brownian Motion (GBM) Model

The GBM Model gained its popularity through its use in the Nobel-Award-winning Black-Scholes stock option pricing analysis.

Propositions and Assumptions

The GBM Model proposes that Stock Growth will largely follow its historical growth patterns; however, this growth may deviate at random. On top of that, the GBM model assumes that some parameters remain constant and assumes that the proportional return of a stock is log-normally distributed.

The following is the GBM Stock Returns Equation:

$$\frac{\Delta S_T}{S_o} = \mu T + \sigma \epsilon \sqrt{T}$$

- S_n is price of the stock at the nth time interval
- T is the number of time intervals after the initial price
- μ is the expected rate of return after one time interval
- σ is the standard deviation of return rates after one time interval
- ϵ is a randomly drawn value from the standard normal distribution

User Wealth Calculation

In real life, many would pair Stock Investments with essentially risk-free growth options like Bonds or Certificate Deposits. If we let r represent this risk-free rate, then that means our possible wealth after investing in a stock can be expressed as:

$$a(K) = wK \cdot \frac{S_T}{S_0} + w(1 - K) \cdot e^{rT}$$

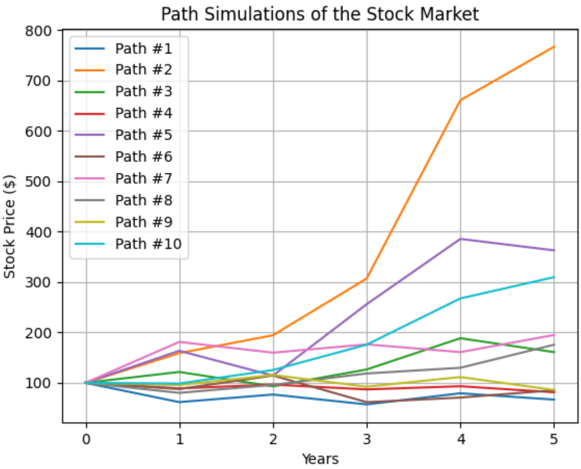
If we treat the GBM Stock Returns Equation as a Stochastic Differential Equation, we can use Ito's Lemma to conclude:

$$S_T = S_0 \cdot e^{(\mu - \frac{\sigma^2}{2})T + \sigma\epsilon\sqrt{T}}$$

This allows us to update our expression for possible wealth:

$$a(K) = w \cdot (K \cdot e^{(\mu - \frac{\sigma^2}{2})T + \sigma\epsilon\sqrt{T}} + (1 - K) \cdot e^{rT})$$

Our Results



Path Simulations of the Stock Market
Path #1
Path #2
Path #3
Path #4
Path #5
Path #6
Path #7
Path #8
Path #9
Path #10
Stock Price ($)
800
700
600
500
400
300
200
100
0
1
2
3
4
5
Years

Kelly Application of the Stock Market

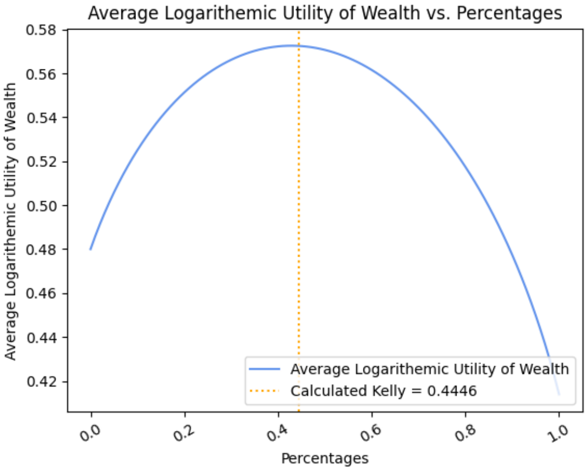
Since Stock Investments have an infinite number of outcomes, Thorpe proposes that we must simplify the problem by creating a "winning" and "losing" scenario. This is done by restricting ϵ to 1 and -1, respectively. This allows us to easily calculate the possible outcomes for the Stock Market. After manipulating the GBM Returns Equation, we found the following formula for the Logarithmic Utility of the Geometric Growth Rate as T approaches infinity.

$$E[U_\infty(K)] = r + K(\mu - r) - \frac{\sigma^2 K^2}{2}$$

By maximizing this equation, we found the Kelly Formula for the Stock Market:

$$K = \frac{\mu - r}{\sigma^2}$$

Our Results


Average Logarithmic Utility of Wealth vs. Percentages
0.58
0.56
0.54
0.52
0.50
0.48
0.46
0.44
0.42
Average Logarithmic Utility of Wealth
0.0
0.2
0.4
0.6
0.8
1.0
Percentages
Average Logarithmic Utility of Wealth
Calculated Kelly = 0.4446

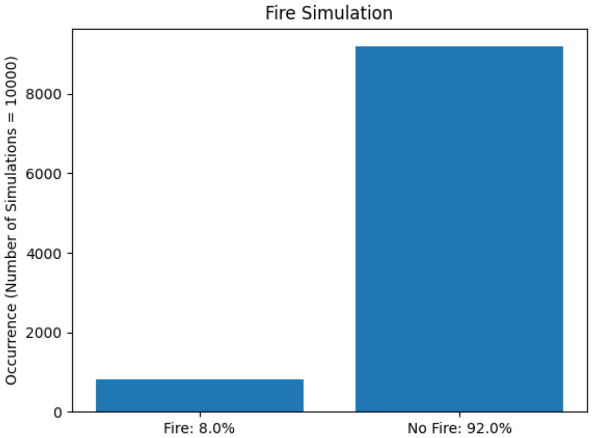
Insurance Model

Our Insurance Model is a Stochastic Process that represents the client's wealth as it changes from month to month. We did this because, in Fire Insurance, the change in the client's wealth could be different if a fire occurs. Since the insurer must compensate for fire damages, which could vary greatly, we simplified the model by assuming that the cost of any fire damage is equal to the house's value. The following expressions show what the client's wealth would be if a fire occurs and if it doesn't:

$$1) W_{1+t} = W_t - hrK - h + hK \qquad 2) W_{1+t} = W_t - hrK$$

- t is the number of Months after the initial Month
- W_n is the client's wealth after the nth Month
- h is the value of the client's house
- r is the insurance rate to the dollar
- K is the coverage that the client bought

Our Results


Fire Simulation
Occurrence (Number of Simulations = 10000)
8000
6000
4000
2000
0
Fire: 8.0%
No Fire: 92.0%

Kelly Application of Fire Insurance

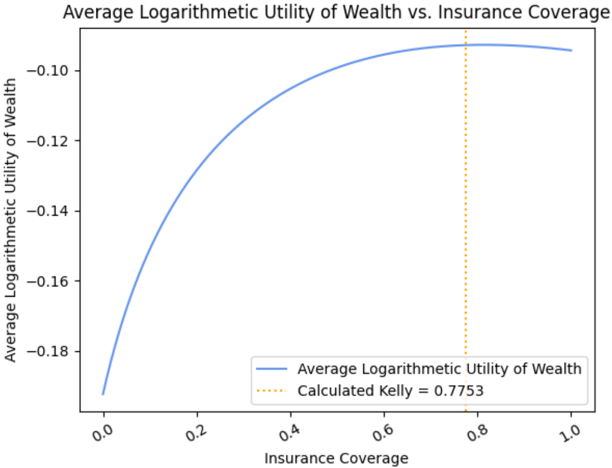
Using our Insurance Model, we can derive a formula for the Average Logarithmic Utility of the Geometric Growth Rate.

$$E[U(K)] = -\ln(w) + p\ln(w - hrK - h + hK) + (1-p)\ln(w - hrK)$$

Due to the Logarithms in this formula, there are values of K where $U(K)$ is undefined. After maximizing this equation, we find:

$$K = \begin{cases} \mathit{Undefined} & 0 \leq r < 1, w \leq hr \\ \mathit{Undefined} & r \geq 1, w \leq h \\ 0 & r \geq 1, w > h \\ 0 & 0 \leq r < 1, h \leq w, w > \frac{hr(1-p)}{r-p} \\ 1 & 0 \leq r < 1, w \geq hr, r < p \\ \frac{w(p-r)+hr(1-p)}{hr(1-r)} & \mathit{Else} \end{cases}$$

Our Results



Average Logarithmetic Utility of Wealth vs. Insurance Coverage
Average Logarithmetic Utility of Wealth
−0.10
−0.12
−0.14
−0.16
−0.18
Insurance Coverage
0.0
0.2
0.4
0.6
0.8
1.0
Average Logarithmetic Utility of Wealth
Calculated Kelly = 0.7753

Final Remarks

Through our code, we recreated what we expect to see from real-world risk-reward situations like the Stock Market and Insurance. From this, we were able to confirm and visually demonstrate the capabilities of the Kelly Criterion to ensure users with an optimal growth plan.

Regarding the Stock Market, our simulations confirm that Thorpe's formula for estimating the Kelly Percentage produces optimal long-term growth. In fact, the risk-free rate fell approximately to zero during the COVID-19 pandemic. This means that the formula would have told the investor to put 100% of their excess wealth into the stock market, and that move would have paid handsomely. We believe that the Kelly Criterion should be a tool used by market analysts.

Regarding Insurance, our simulations indicate that the Kelly Criterion could advise consumers not to fully insure their homes. This could potentially save money for consumers.

Future Work

In our research, our models used a lot of simplifying assumptions that could overlook some important factors.

In the Stock Market, we only looked at one stock index and we assumed that its growth variance remained constant over time. Some future work that focuses on this could try to apply the Kelly Criterion to a more complicated model and involve more stock-trading scenarios. For example, a portfolio consisting of a risk-free investment and options on shares of a highly-traded company, such as Apple.

Regarding Insurance, we only looked at one type of insurance and we simplified the scenario. A future work that focuses on this could try to apply the Kelly Criterion to a different kind of insurance. Another potential avenue of research would be to include various levels of fire severity in the Insurance Model. This will extend our Insurance Model further and could result in a modified Kelly Formula.

Acknowledgements

We would like to thank our faculty mentor, Dr. Douglas Eckley, for his guidance and support throughout the semester, and the MEGL team for the opportunity to present our research.

References

- Carta, A., Conversano, C. (2020). Practical Implementation of the Kelly Criterion: Optimal Growth Rate, Number of Trades, and Rebalancing Frequency for Equity Portfolios. Frontiers in Applied Mathematics and Statistics, 6. https://doi.org/10.3389/fams.2020.577050
- Desrosiers, M. (2012). How individuals purchase insurance: Going beyond expected utility theory. In Casualty Actuarial Society E-Forum (Vol. 2, No. 1, pp. 1-18).
- Hankin, P. (2018, June 10). Insurance and the Kelly Criterion. Paul's Blog. https://blog.paulhankin.net/kellycriterion/
- Kelly criterion. (2023, November 20). Wikipedia. https://en.wikipedia.org/w/index.php?title=Kelly$_c$*riterion*

References

- Kenton, W. (2022, May 9). Kelly Criterion: Definition, How Formula Works, History, and Goals. Investopedia. https://www.investopedia.com/terms/k/kellycriterion.asp
- Khanna, N. (2016). The Kelly criterion and the stock market.
- Peterson, M. (2019). The St. Petersburg Paradox. Plato.stanford.edu. https://plato.stanford.edu/archives/fall2023/entries/paradox-stpetersburg/
- Reddy, K., Clinton, V. (2016). Simulating stock prices using geometric Brownian motion: Evidence from Australian companies. Australasian Accounting, Business and Finance Journal, 10(3), 23-47.
- Thorp, E. O. (2008). The Kelly criterion in blackjack sports betting, and the stock market. In Handbook of asset and liability management (pp. 385-428). North-Holland.