The Effects of Surface Tension on Lowest-Energy Floating Configuration of 3D-Printed Objects

Daniel Horvath, Max Werkheiser, Brandon Barreto-Rosa, Patrick Bishop, Mark Brant, Dr. Dan Anderson, Dr. Evelyn Sander

Mason Experimental Geometry Lab

GEORGE MASON UNIVERSITY

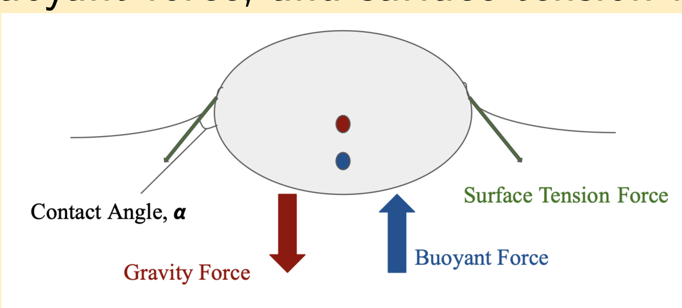
Dec 1st, 2023

Introduction

We investigate lowest-energy floating configurations of 3D-printed objects with a polygonal cross-section of uniform density along an axis involving gravitational, buoyant, and surface tension forces. We extend Archimedes Principle to include fluid volume of the meniscus to account for surface tension in the force balance equation.

Forces Acting on a Floating Object

The orientation of a stationary object satisfying the parameters above may be characterized by the balance of gravitational force, buoyant force, and surface tension force.



Mathematically, this may be written:

$$0 = \vec{F_g} + \vec{F_p} + \vec{F_T}$$

With forces defined below:

$$\vec{F_g} = M_{obj}\vec{g}$$

$$\vec{F_p} = \ell\rho_f g\{\frac{1}{2}(y_R{}^2 - y_L{}^2), A_{sub} - A_{\text{meniscus}}\}$$

$$\vec{F_T} = \ell\gamma_L\vec{t_L} + \ell\gamma_R\vec{t_R}$$

Potential Energy of the System

The potential energy of a system may be determined from the energy corresponding to each of the above forces. Consider some constant point on the boundary of the object, $\vec{x}$. Then call some angle between $\vec{x}$ and the waterline θ. Equilibrium floating configurations occur at an angle θ, for which $U(\theta)$ has minima.

$$U(\theta) = PE_G + PE_B(\theta) + \Delta E(\theta)$$

The definitions for each of the above three terms are still being developed.

Bond Number and Relevance of Meniscus

The Bond Number, B_0, is a dimensionless parameter used to compare the magnitude of the gravitational and surface tension forces experienced by a floating object at a phase interface.

$$B_0 = \frac{\Delta\rho\ell^2 g}{\gamma}$$

where $\Delta\rho = \rho_{fluid}\text{-}\rho_{object}$, ℓ is the length scale of the object, g is acceleration due to gravity, and γ is the surface tension constant.

Archimedes Principle and Surface Tension

The buoyant force acting on an object floating in a fluid is equal and opposite to the force of gravity acting on the object. This is written:

$$M_{obj}g = \rho_f\ell A_{sub}g.$$

If the bond number is sufficiently large, then surface tension becomes negligible. Otherwise, the fluid displaced from the waterline significantly contributes to the buoyant force and the area of the meniscus must be accounted for in A_{sub}.

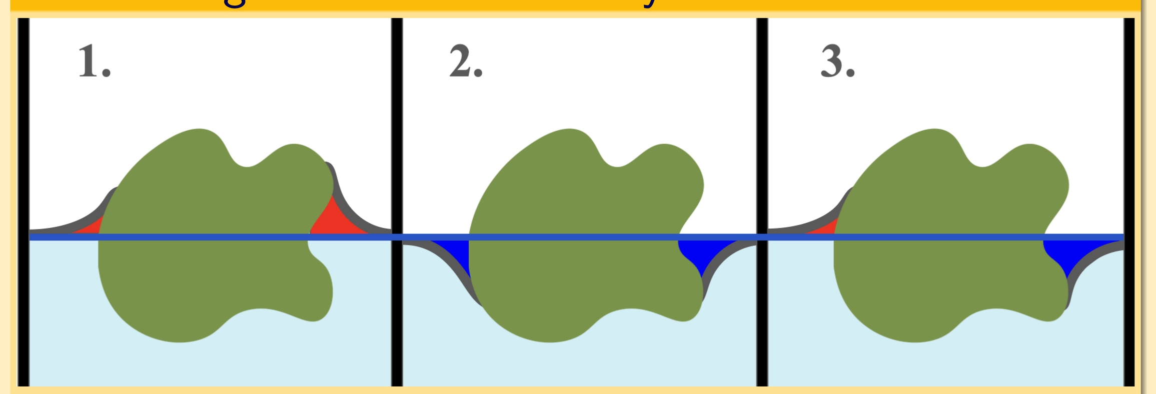
Shape of the Meniscus

Contact Angle

Equilibrium contact angle, α, is intrinsic to a system of a solid intersecting a phase interface at 90°. That is, contact angle is measured such that the object is essentially a vertical wall. α is determined from reliable experimental data. The expected direction of a meniscus can therefore be predicted from α relative to 90 °. If $\alpha < 90°$, the meniscus is upward. If $\alpha > 90°$, the meniscus is downward.

The lowest-energy, or equilibrium, floating configuration of an object is therefore one in which the contact angle is equal to α at all waterline contact points.

Three Possible Cases of Menisci at an Object Intersecting the Phase Boundary at Two Points

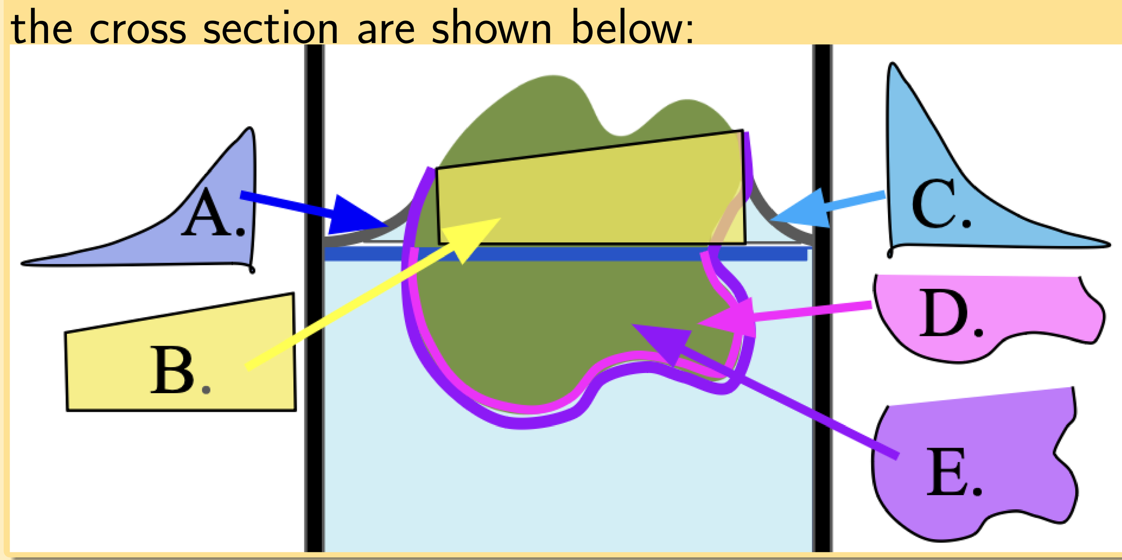


Blue: Increases amount of water displaced, stronger buoyant force

Red: Decreases amount of water displaced, weaker buoyant force

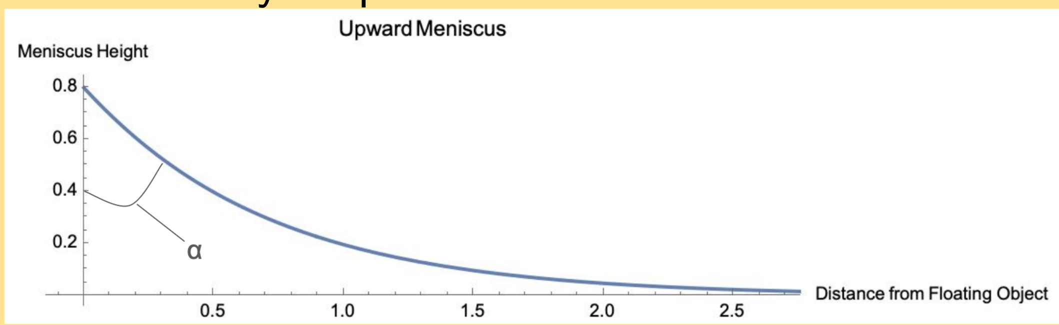
Evaluating Cross Sectional Area of Displaced Fluid: The Trapezoid Method

For A_{sub} computation, five relevant bounded regions of the cross section are shown below:



Function Describing the Meniscus

An ODE for the height of a liquid surface may be derived from the normal vector at the surface of the liquid, $\hat{n}$ and the pressure of a static fluid, p(h) [2]. For the case where $y_0 = 0.8$ and $d = \frac{1}{\sqrt{2}}$, the meniscus may be plotted:



Differential Equation for Meniscus Height

$$1 = \frac{1}{\sqrt{1 + (\frac{dh}{dx}^2)}} + \frac{h^2}{2d^2},\ d^2 = \frac{\gamma}{\rho_f g}$$

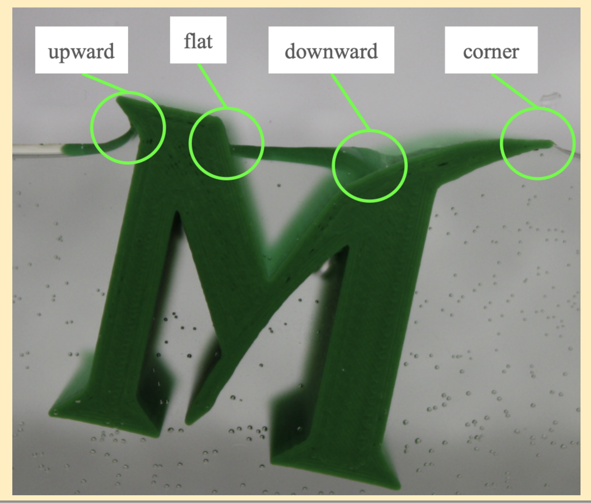
Meniscus Height Function

$$x(h = y_0) = 0, \left.\frac{dx}{dh}\right|_{x=0} = -tan(\alpha)$$

$$\frac{x(h)}{d} = \text{arccosh}(\frac{2d}{h}) - \text{arccosh}(\frac{2d}{y_0}) + \sqrt{4 - \frac{y_0^2}{d^2}} - \sqrt{4 - \frac{h^2}{d^2}}$$

Case Study: Flotation of the Mason "M"

The print of the Mason "M" floating in water on the right depicts an example of a case where many different kinds of menisci may occur at the equilibrium floating configuration.



Future Work

1. Implement these principles in existing code to find stable floating orientations for objects experiencing non-negligible surface tension
2. Adapt the model to predict floating configuration for an object that intersects the phase interface more than two times
3. Further investigate cases when menisci form at sharp corners
4. Further investigate the possibility of multiple stable configurations existing for each rotational orientation
5. Apply these findings to 3D-printed objects that float on top of a liquid without breaking its surface, as seen in certain insect species

Acknowledgments

We would like to extend our gratitude to our faculty mentors, Dr. Sander and Dr. Anderson, and our graduate student mentors, Brandon Barreto-Rosa, Patrick Bishop, and Mark Brant. We also thank the MakerLab and MEGL.

References

1. Anderson et al., "Mathematics of Floating 3D Printed Objects," AMS Proceedings of Symposia in Applied Mathematics, 2022.
2. G.K. Batchelor, Fluid Dynamics (Cambridge University Press, 1967).
3. D. Vella, L. Mahadevan, The "Cheerios effect." Am. J. Phys., 2005. doi: 10.1119/1.1898523