# The Effects of Surface Tension on Lowest-Energy Floating Configuration of 3D-Printed Objects

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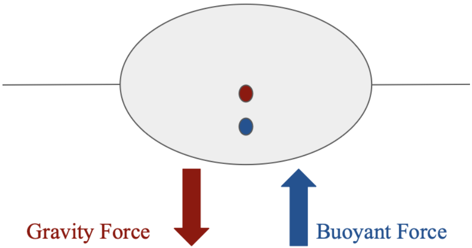
# Introduction

- In this study, we consider an object of uniform density along an axis of length $\ell$ with a polygonal cross section floating at a fluid-air interface
- We can describe the potential energy of the system as a function of the angle, $\theta$. $\theta$ is the angle of object's rotation from a fixed orientation defined as $\theta = 0$
- The system is at equilibrium when the potential energy is minimized

# Previously: No Consideration of Surface Tension Effects

- Assumes that the object is experiencing the force of gravity and buoyant force by Archimedes Principle
- For a given shape and density ratio, predicts potential energy as a function of $\theta$
- For a given shape, predicts bifurcation diagram for number of energy minima as a function of density ratio

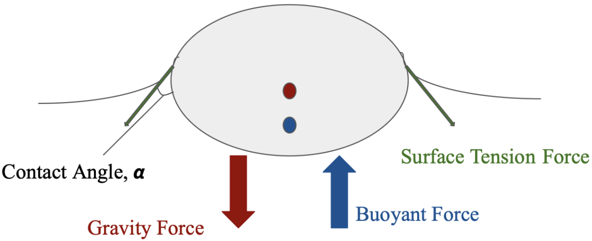


# Currently: Consideration of Surface Tension Effects

- We aim to account for surface tension in the potential energy calculation

- Bond Number, $B_0$, is a dimensionless, quantitative measure of the ratio of gravitational force and surface tension force

  - $\mathrm{B}_0 = \frac{\Delta\rho L^2 g}{\gamma}$



# Three Forcesketeers: Relevant Forces

## Gravitational Force

$$\vec{F_g} = M_{obj}\vec{g}$$

## Pressure/Buoyant Force

Let $\hat{n}_{obj}$ = unit normal force, $\partial\Omega_{sub}$ = the boundary submerged region, and $p_A$ = atmospheric pressure. The pressure force on the object is:

$$\vec{F_p} = -\ell \int_{\partial\Omega_{sub}} (p(s) - p_A)\hat{n}_{obj}(s)ds = \ell\rho_f g\left\{\frac{1}{2}({y_R}^2 - {y_L}^2), A_{sub} - A_{\text{trap}}\right\}$$

## Surface Tension Force

The surface tension force is tangent to the meniscus at the point where the object intersects the surface of water.

$$\vec{F_T} = \ell\gamma_L\vec{t_L} + \ell\gamma_R\vec{t_R}$$

# Archimedes Principle and Surface Tension

## Archimedes Principle

The buoyant force acting on an object floating in a fluid is equal and opposite to the force of gravity acting on the object. This is written:

$$M_{obj} g = \rho_f \ell A_{sub} g.$$

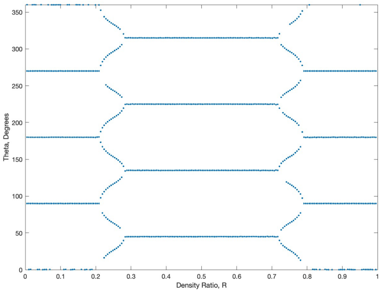
Where $M_{obj}$ is the mass of the object, g is acceleration due to gravity, $\rho_f$ is the fluid density, and $A_{sub}$ is the area of the displaced fluid.

- If $B_0$ is sufficiently large, surface tension is negligible and $A_{sub}$ is equivalent to the area of the object below the waterline.(iceberg case)
- Otherwise, the fluid displaced from the waterline significantly contributes to the buoyant force and the area of the meniscus must be accounted for in $A_{sub}$

# Potential Energy

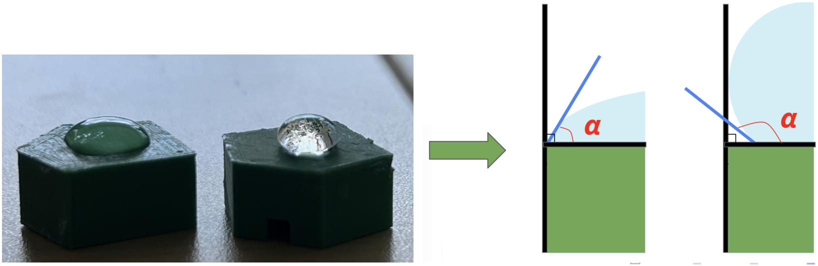
- Each force acting on the object has a corresponding potential energy
- Equilibrium floating configurations occur at angles $\theta$, for which $U(\theta)$ has minima $U(\theta) = PE_G + PE_B(\theta) + PE_T(\theta)$
- We intend to use this new description of U($\theta$) to predict equilibrium angles as a function of the density ratio R $= \frac{\rho_{obj}}{\rho_{fluid}}$, and Bond number $B_0$

**Bifurcation Diagram for an Object with a Square Cross Section w/o Surface Tension**

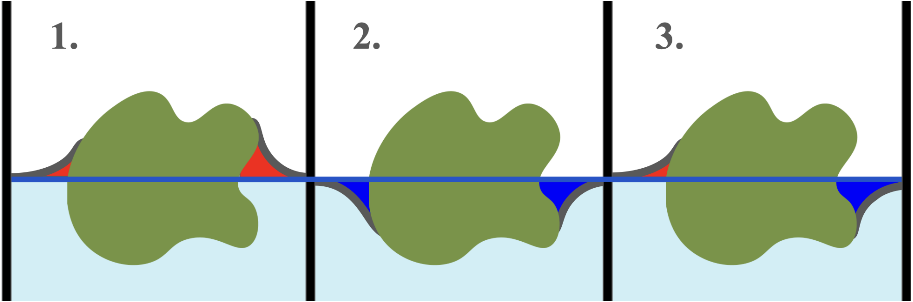


# Contact Angle

- Equilibrium contact angle, $\alpha$, is intrinsic to a system of a solid intersecting a phase interface
- The expected direction of a meniscus can be predicted from $\alpha$
- Non-equilibrium contact angles may be achieved by applying external force that overcomes the force of surface tension
- The lowest-energy, or equilibrium, floating configuration of an object is one which satisfies $\alpha$ at all waterline contact points



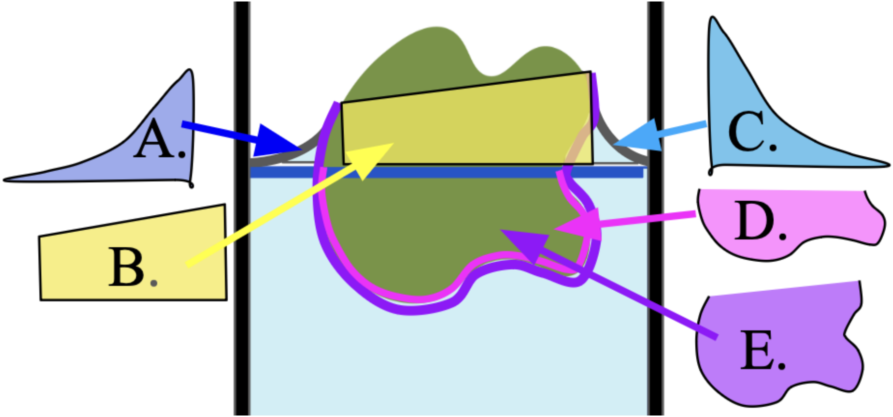
# Possible Menisci Cases for a Floating Object




Blue: Increases amount of water displaced, stronger buoyant force
Red: Decreases amount of water displaced, weaker buoyant force

# Evaluating Cross Sectional Area of Displaced Fluid: The Trapezoid Method

For $A_{sub}$ computation, five relevant bounded regions of the cross section are shown below:



# Function for Meniscus Height

An ODE for the height of a liquid surface may be derived from (a) the normal vector at the surface of the liquid, $\hat{n}$ and (b) the pressure of a static fluid, p(h).

## Derivation

We have initial conditions:

$$x(h = y_0) = 0, \left.\frac{dx}{dh}\right|_{x=0} = -tan(\alpha)$$

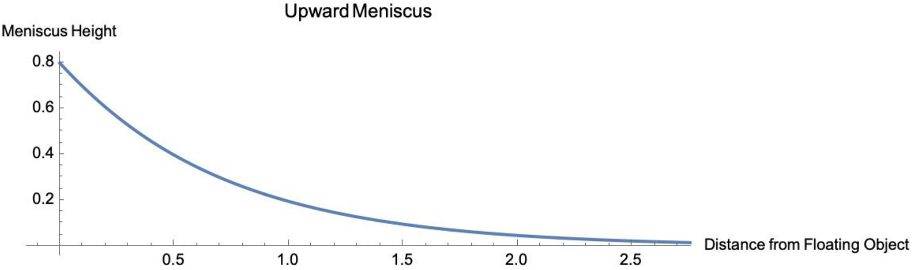
for the ODE:

$$1 = \frac{1}{\sqrt{1+(\frac{dh}{dx}^2)}} + \frac{h^2}{2d^2}, \text{ where } d^2 = \frac{\gamma}{\rho_f g}$$

We can solve the ODE to get a function describing the position, x, along the waterline as a function of height of the meniscus, h.

$$\frac{x(h)}{d} = \text{arccosh}(\frac{2d}{h}) - \text{arccosh}(\frac{2d}{y_0}) + \sqrt{4 - \frac{{y_0}^2}{d^2}} - \sqrt{4 - \frac{h^2}{d^2}}$$

For the case where $y_0 = 0.8$ and $d = \frac{1}{\sqrt{2}}$, the meniscus may be plotted as



# Future Work

- Implement these principles in existing code to find stable floating orientations for objects experiencing non-negligible surface tension
- Adapt the model to predict floating configuration for an object that intersects the phase interface more than two times
- Further investigate cases when menisci form at sharp corners
- Further investigate the possibility of multiple stable configurations existing for each rotational orientation
- Apply these findings to 3D-printed objects that float on top of a liquid without breaking its surface, as seen in certain insect species

# References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# Appendix

## Normal Vector from Liquid Surface

$$\hat{n} = \frac{1}{\sqrt{1 + (\frac{dh}{dx})^2}} \left\{ -\frac{dh}{dx}, 1 \right\}$$

## Pressure of Liquid at Interface: Young-Laplace Equation

$$p(h) = p_A + \gamma \Delta \cdot \hat{n}$$

where $p_A$ is air pressure and is assumed constant.