The Dirichlet Problem on Select Subsets of $\mathbb{R}^2$.

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There can be but one opinion as to the beauty and utility of this analysis of Laplace; but the manner in which it has been hitherto presented has seemed repulsive to the ablest mathematicians, and difficult to ordinary mathematical students.
- Lord Kelvin and Peter Trait, *Treatise on Natural Philosophy*, 1879

The Dirichlet Problem [5][6]

Definition

A real-valued function u on an open subset $\Omega \subseteq \mathbb{R}^n$ is harmonic if it is

1. twice continuously differentiable, and
2. the Laplacian of u, defined $\Delta u = \partial^2 u/\partial x_1^2 + ... + \partial^2 u/\partial x_n^2$, is 0 throughout Ω.

The Dirichlet problems asks for a given bounded region Ω, "Does there exist such a function u as, defined above, that is continuous within $\bar{\Omega}$ and agrees with a given function R on the boundary?"

Its main applications are in the physics of heat flow, electrostatics, and other fields.

Examples on the Unit Disk D in $\mathbb{R}^2$ [1]

Given the following data, find a function that solves the Dirichlet Problem.

Example 1.1

Let $R_1(x, y) = x(x^2 + y^2)$.
$u_1(x, y) = x$ is a solution since on the unit circle, $x^2 + y^2 = 1$.

Example 1.2

Let $R_2(x, y) = 1/(5 + 3x)$.
Then the solution is

$$u_2(x, y) = \frac{9 - x^2 - y^2}{36 + 24x + 4(x^2 + y^2)}.$$

In general, the solution will be very complicated.

Conformal Maps [5]

Let Ω be a domain in the plane such that there exists a conformal map $\varphi : \Omega \to D$ with $\varphi(\Omega) = D$, if we find a solution u to the Dirichlet problem to boundary data $R \circ \varphi^{-1}$ (on the unit disk), where R is the original boundary data function, $\varphi \circ u$ will still be harmonic, and a solution to the Dirichlet problem with the original parameters.

This is why we are generally working with disks, since other Dirichlet problems can be translated to it.

General Solution in the Disk [5][6]

On disks centered at the origin, we can represent the boundary function by the function T, where $T(\theta)$ is the value of our data function on the boundary circle with radius R at angle θ. Then the solution to the Dirichlet Problem is found in general by the Poisson Integral:

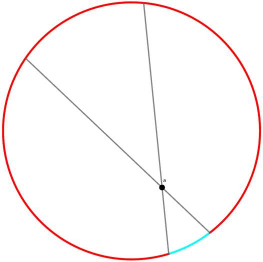
$$u(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \alpha)} \right] T(\theta)\, d\theta,$$

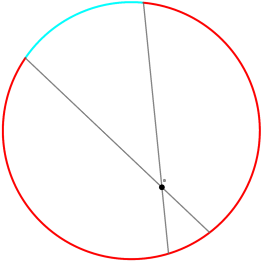
where any point in the disk $a = re^{i\alpha}, (r < R)$.

But for more well-behaved data functions, we can find more elegant methods to finding the solution.

Schwarz Interpretation [5]

On disks in particular there is a visual interpretation of the Poisson Integral. Take the boundary data and reflect it across a given point a. A weighted average of the points on the reflected circle will equal the value of the solution to the Poisson Integral.





Complex Analytic Approach [1][6]

- Consider the boundary data given by a rational function $R(x, y)$ on the boundary of the unit disk ∂D.
- Objective: It is known that the real part of an analytic function is harmonic. We wish to find $H(z)$ analytic on the disk D such that the real part of H equals R on the boundary ∂D.
- Use the change of coordinates $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$ to obtain a function of one complex variable

$$h(z) = R((z + 1/z)/2, (z - 1/z)/2i). \tag{1}$$

- h is a rational function continuous on ∂D and equal to R on ∂D.

Complex Analytic Approach (contd.)

- We can decompose h into a sum of a polynomial and a rational function in z: $h(z) = p(z) + s(z)$.
- As a polynomial, $p(z)$ is already analytic. However $s(z)$ may have poles inside the disk, and so requires modification by reflecting the poles outside the disk.
- For each term $k_m(z) = a/(z-c)^{n_m}$ in $s(z)$ where $a, c \in \mathbb{C}$, $n \in \mathbb{Z}^+$, and $|c| < 1$, replace with

$$K(z) = \overline{k(1/\bar{z})} = \frac{\bar{a} z^n}{(1 - \bar{c} z)^n} \tag{2}$$

Complex Analytic Approach (contd.)

- Note that the real parts of $K(z)$ and $k(z)$ are equal on the boundary, so the values of their real parts on the boundary stay the same.
- Define the function $H(z) = p(z) + S(z)$ where $S(z)$ is obtained from $s(z)$ by replacing each term $k(z)$ with $K(z)$ as described before.
- Our solution $u = \text{Re } H$.
- One can show using this method that if R is a polynomial, so is u.

Linear Algebraic Approach - Fischer's Lemma [2]

Theorem (Fischer's Lemma, 1917)

Consider the operator $L\colon \mathbb{P}_m[x_1, \ldots, x_n] \longrightarrow \mathbb{P}_m[x_1, \ldots, x_n]$ *defined by* $L(f) = \Delta(q \cdot f)$, *where* $q(x_1, \ldots, x_n) = 1 - \sum_{k=1}^{n} x_k^2/r_k^2$ *for* $r_k > 0$. *Then* L *is linear, degree-preserving, and a bijection from the real algebra of polynomial functions from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *onto itself.*

- $\mathbb{P}_m[x_1, \ldots, x_n]$ is the space of polynomials of (multi)degree at most m in variables $x_1, \ldots, x_n$

Allows us to construct an algebraic solution to the Dirichlet problem in the case of polynomial boundary data when the domain Ω is the interior of an ellipse of the equation $q(x_1, \ldots, x_n) = 0$.

Linear Algebraic Approach [2][4]

Theorem (Gonzales 2014)

Given $f \in \mathbb{P}_m$, there exists a unique solution $u \in \mathbb{P}_m$ to the Dirichlet problem given by

$$u = f - q \cdot L^{-1}(\Delta(f))$$

Moreover, the linearity of L lets us compute solutions to the Dirichlet problem using the tools of linear algebra; to do this, we need to construct a matrix representation of L under some ordered basis of $\mathbb{P}_m$ in variables $x_1, \ldots, x_n$.

Example 2

Suppose we have a Dirichlet problem over the unit disk with polynomial data given by $f \in \mathbb{P}_4[x, y]$ so that $q(x, y) = x^2 + y^2 - 1$; choose the ordered basis of $\mathbb{P}_4[x, y]$ given by $\mathcal{B} = \left\{1, x, y, x^2, xy, y^2\right\}$. Then we have the following basis for $L(f) = \Delta(qf)$

$$\begin{aligned} L(\mathcal{B}) &= \left\{\Delta(q), \Delta(qx), \Delta(qy), \Delta(qx^2), \Delta(qxy), \Delta(qy^2)\right\} \\ &= \left\{4, 8x, 8y, 14x^2 + 2y^2 - 2, 12xy, 2x^2 + 14y^2 - 2\right\} \end{aligned}$$

Hence, the matrix representaton of $L\colon \mathbb{P}_4[x, y] \longrightarrow \mathbb{P}_4[x, y]$ is given by

$$[\mathrm{L}] = \begin{bmatrix} 4 & 0 & 0 & -2 & 0 & -2 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14 & 0 & 2 \\ 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 2 & 0 & 14 \end{bmatrix}$$

Linear Algebraic Approach

Example 2.1

Now suppose our boundary data is given by the polynomial $f(x, y) = y^2$. Note that $\Delta(y^2) = 2$ which has the vector representation $(2, 0, 0, 0, 0, 0)^t$ so that the solution to this Dirichlet problem is given by

$$\begin{aligned} u &= f - q \cdot L^{-1}(\Delta(y^2)) \\ &= y^2 - \frac{1}{2}(x^2 + y^2 - 1) \end{aligned}$$

where $L^{-1}(\Delta(y^2))$ is found by computing the matrix-vector product $[L]^{-1}(2, 0, 0, 0, 0, 0)^t$.

Extending L to the case of rational boundary data

We are able to extend the use of L to the case of rational boundary data by introducing a restriction to rings of rational functions with a fixed denominator polynomial. However, we find that L, when applied to this restriction, is not necessarily one-to-one. In particular, for $L(P/Q) = \widetilde{P}/\widetilde{Q}$, it is the case that

$$\deg \widetilde{P} \leq \deg P + 2\deg Q$$
$$\deg \widetilde{Q} = 3\deg Q$$

Homogeneous polynomial boundary data

In the case of homogeneous polynomial data, we are able to directly compute a solution to the Dirichlet problem on a disk by way of harmonic decomposition. That is, every $p \in \mathcal{P}_m(\mathbb{R})$ can be uniquely written in the form

$$p = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} |x|^{2k} p_{m-2k} \tag{3}$$

where $p_k \in \mathcal{H}_k(\mathbb{R})$ (space of real-valued homogeneous harmonic polynomials of degree k) for every k. It then follows that, if p is the boundary data function in a Dirichlet problem, then the solution to said Dirichlet problem, u, is given by

$$u = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} p_{m-2k} \tag{4}$$

Restricting the domain of L to $\mathcal{H}_m$ and $\mathcal{P}_m$

When restricting the domain of L to homogeneous polynomials, we are able to preserve the properties stated by Fischer's lemma. In particular, we can show directly that $\mathcal{H}_m(\mathbb{R})$ is L-invariant with eigenvalue $4m + 1$ and, building on this, we find that

$$L\colon \bigoplus_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathcal{P}_{m-2k}(\mathbb{R}) \longrightarrow \bigoplus_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathcal{P}_{m-2k}(\mathbb{R}) \tag{5}$$

is a linear, degree-preserving bijection from $\bigoplus_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathcal{P}_{m-2k}(\mathbb{R})$ onto itself.

Poisson integral as discrete sum at the center of the disk

- In the polynomial data case, we can get a discrete sum from the Poisson integral formula for the value at the origin using the induction formulae for products of sines and cosines.

$$\int \cos^m(\theta)\sin^n(\theta)d\theta = -\frac{\cos^{m+1}(\theta)\sin^{n-1}(\theta)}{n+m} + \frac{n-1}{n+m}\int \cos^m(\theta)\sin^{n-2}(\theta)d\theta, \quad (6)$$

$$\int \cos^m(\theta)\sin^n(\theta)d\theta = \frac{\cos^{m-1}(\theta)\sin^{n+1}(\theta)}{n+m} + \frac{m-1}{n+m}\int \cos^{m-2}(\theta)\sin^n(\theta)d\theta. \quad (7)$$

Interpolation

It is known that for polynomial $p(x, y)$, the solution to the Dirichlet problem can be represented as the real part of an analytic function u of the form

$$u(z) = d_0 + \frac{1}{2} \sum_{k=0}^{m} \left(c_k z^k + \bar{c_k} \bar{z}^k \right).$$

We can then set up a system of $2m + 1$ linear equations in the unknowns $(d_0, c_1, \overline{c_1}, \ldots, c_{2m}, \overline{c_{2m}})$ by letting z take the values $z_0, \ldots, z_{2m}$, the roots of unity of order $(2m + 1)$:

$$p(x_i, y_i) = u(z_i) = d_0 + \frac{1}{2} \sum_{k=0}^{m} \left(c_k z_i^k + \bar{c_k} \bar{z}_i^k \right).$$

The coefficient matrix of the system is (the transpose of) a Vandermonde matrix, as seen below.

Interpolation (contd.)

Example 3.1

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & z_1^1 & \overline{z_1}^1 & \dots & z_1^m & \overline{z_1}^m \\ 1 & z_2^1 & \overline{z_2}^1 & \dots & z_2^m & \overline{z_2}^m \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & z_{2m}^1 & \overline{z_{2m}}^1 & \dots & z_{2m}^m & \overline{z_{2m}}^m \end{pmatrix} \begin{pmatrix} d_0 \\ c_1 \\ \overline{c_1} \\ \vdots \\ \overline{c_{2m}} \end{pmatrix} = \begin{pmatrix} u(1) \\ u(z_1) \\ u(z_2) \\ \vdots \\ u(z_{2m}) \end{pmatrix}$$

Since we know the values $u(z_i)$ because the points z_i are on the boundary, we can determine the coefficients, and obtain u.

We are considering a similar approach finding the coefficients of $V(z) = z^m u(z)$, which on the unit circle, is a polynomial in z. The coefficients of $V(z)$ can be used to retrieve u. We are also considering using Lagrange interpolation for V.

Further direction

- Is it possible to obtain a discrete sum version of the Poisson integral formula for points other than the origin using interpolation?

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