Implementing the Polar Method

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Introduction

For our project, we used probability and statistical theory to analyze the Polar method, which is a method for generating pairs of independent random standard normal variables. We also explain how the Polar method is derived from the Box-Muller transform and discuss the limitations and efficiencies of each method. The generated values from the Polar method are used to simulate sunlight which is assumed to influence the growth of grapes needed for wine production. We analyzed our model using some economic theories in order to guide the investment strategy and simulate the potential profit or loss that could be generated from the investment in the vineyard.

Pseudorandom Generation

Definition (Inverse Transform Method)

Let $F_X(x) = P(X \leq x)$ be the cumulative distribution function for the random variable X. If $U \sim \text{Unif}(0,1)$, then $F_X^{-1}(U)$ has the same cululative distribution function F_X. In other words, the generalized inverse function $F_X^{-1}(u) = \inf\{x : F(x) \geq u\}$ for all $u \in (0,1)$ has the CDF F_X.

Definition (Box-Muller Transform)

Step 1. Generate U_1 and U_2 from $U \sim U(0,1)$
Step 2. Let $z_0 = \sqrt{-2ln(U_1)}\, cos(2\pi U_2)$ and $z_1 = \sqrt{-2ln(U_1)}\, sin(2\pi U_2)$
Step 3. Return the standard independent normal random variables within the coordinate pair (z_0, z_1)

Definition (Marsaglia Polar Method)

Step 1. Generate U_1 and U_2 from $U \sim U(0,1)$
Step 2. Let $V_1 = 2U_1 - 1$ and $V_2 = 2U_2 - 1$
Step 3. If $S = V_1^2 + V_2^2 > 1$, then go back to Step 1.
Step 4. Return the standard independent normal random variables $X = V_1\sqrt{\frac{-2\ln(S)}{S}}$ and $Y = V_2\sqrt{\frac{-2\ln(S)}{S}}$

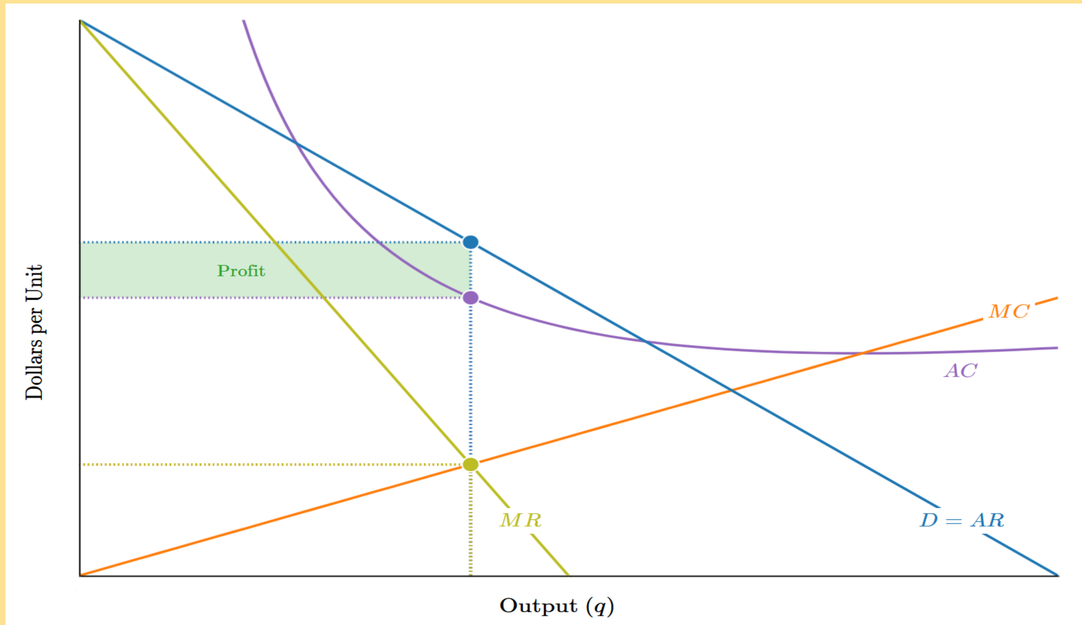
Definition (Relationship with Chi-Square Distribution)

Given a sum of iid standard normal variables $\sum_{i=1}^{k} X_i$ obtained via the Polar Method, we can obtain a Chi-square distribution of arbitrary degree k, given by

$$f(x;k) = \begin{cases} \frac{x^{\frac{k}{2}-1} e^{\frac{-x}{2}}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}, & \text{if } x > 0; \\ 0, & \textit{otherwise} \end{cases}$$

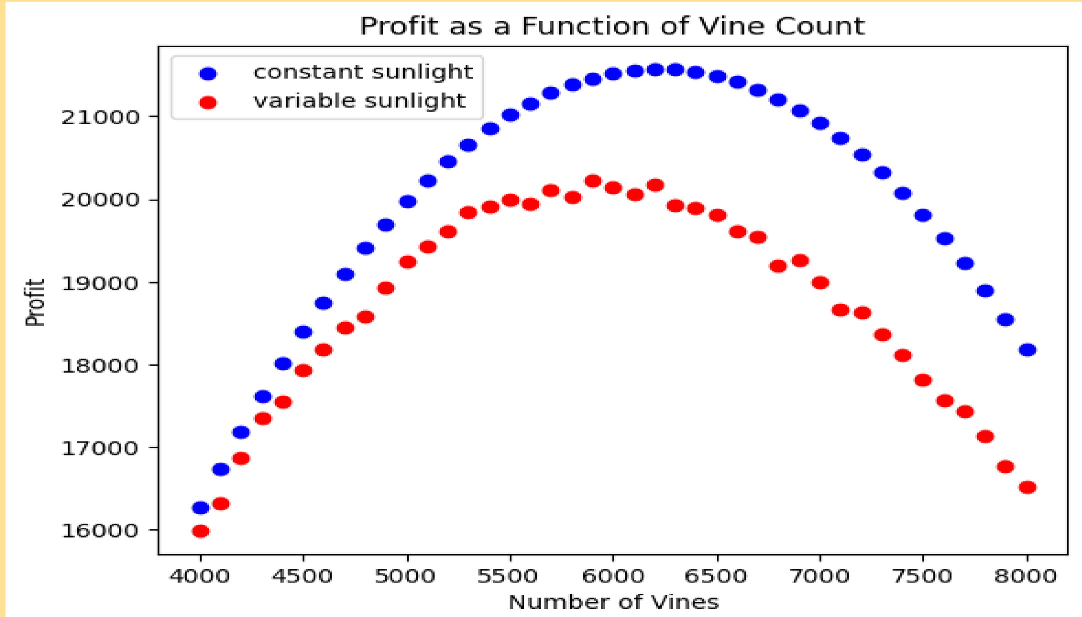
Process

Once the polar method was utilized to generate standard normal numbers for sunlight, we can then calculate the amount of bottles produced. However, there are some factors that need to be considered, such as the typical spoilage rate at about 10%, the expected variable costs to produce one bottle, and the effect of cloudy days on the progress within the vineyard. The revenue is determined by the amount of bottles and the price, while profit is the total production revenue after accounting for all the costs.



Is The Vineyard a Good Investment?

To answer this question, we must compare deterministic sunlight (equal to 6 hours per day) to randomly generated simulations of sunlight. We find that using 10,000 simulations of sunlight had caused the maximum expected profit to fall to \$20,143, based on 5,990 vines planted. We were thus able to still make a profit, but it was somewhat lower than expected, especially in comparison to the expected profit calculated from the constant sunlight. We thus will plant less plants than we would have predicted using the constant sunlight values. The difference can be explained using Jensen's inequality.



Variance and Risk

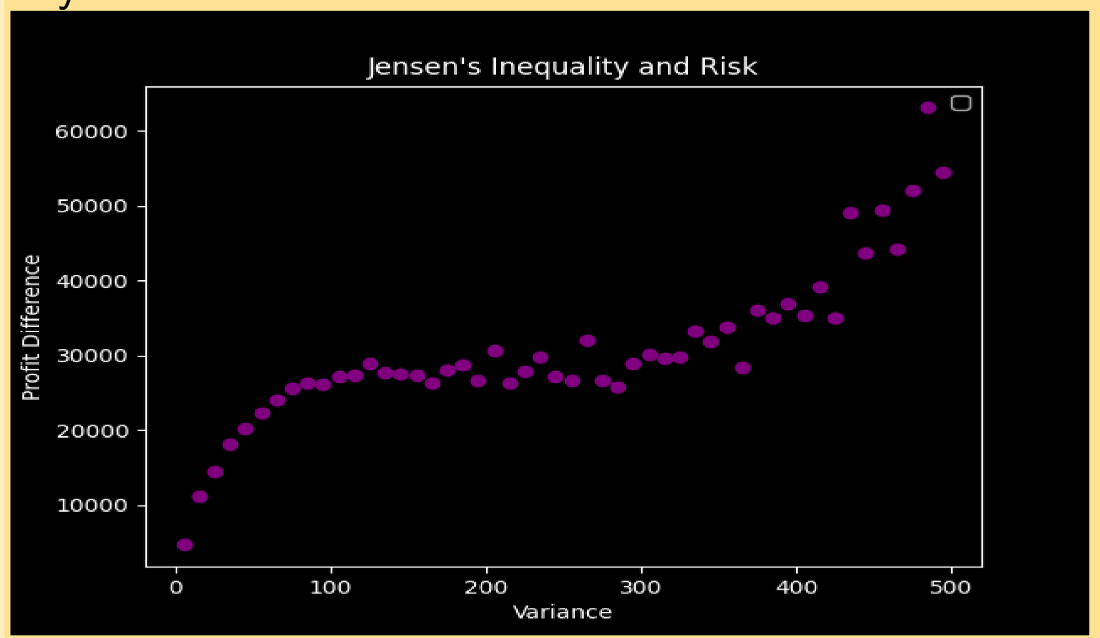
Definition (Jensen's Inequality)

For a concave function, the mean transform of an observation mean $(f(x))$ is always less than the transform of the mean observation $f(mean(x))$ Jensen's inequality: $E[f(X)] \leq f(E[X])$

Jensen's inequality suggests that the mean of the profits will always be smaller than or equal to the profit of the mean outcome. Consequently, a vineyard with constant sunlight hours k yields a greater expected profit than is possible should the hours of sunlight be determined by $X \sim \mathcal{N}(\mu = k, \sigma^2)$. Market actors have to account for this variance σ^2 when pricing products.

Jensen's Inequality and Risk Aversion

The payout from wine harvests is defined by a concave loss function, and necessarily $f(E[X]) \leq E[f(X)]$. Rational actors will reduce production, and may even exit the market altogether should uncertainty increase beyond what their risk tolerance would allow for.



Conclusions and Future Work

The vineyard was found to be profitable and we were able to find the number of vines (5,990) that would give the max profit. Due to climate change, there will be an increase in hours of sun throughout the vineyard which will lead to more restricted production (i.e. higher prices, and lower profits). This will amount to neither the vineyard owners nor the customers benefiting from the sales within the vineyard. Future work would be to create a variable for global warming or simulating this vineyard within different regions.

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