Foliations on Surfaces and Such

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Introduction: Surfaces

Let Σ be a two dimensional topological space covered by open sets U_α for some index α. If there exist homeomorphisms

$$\phi_\alpha : U_\alpha \to \mathbb{R}^2$$

for every α, then Σ is called a **surface**.

Theorem

Poincaré-Kneser: Let Σ be a compact, connected surface. Then Σ admits a regular foliation if and only if its Euler characteristic is zero.

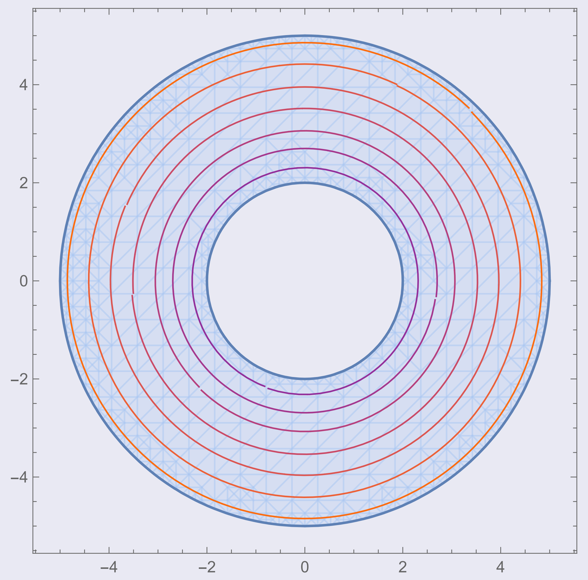
As a consequence of the Poincaré-Kneser Theorem, we will only be looking at compact surfaces for which the Euler Characteristic is zero:

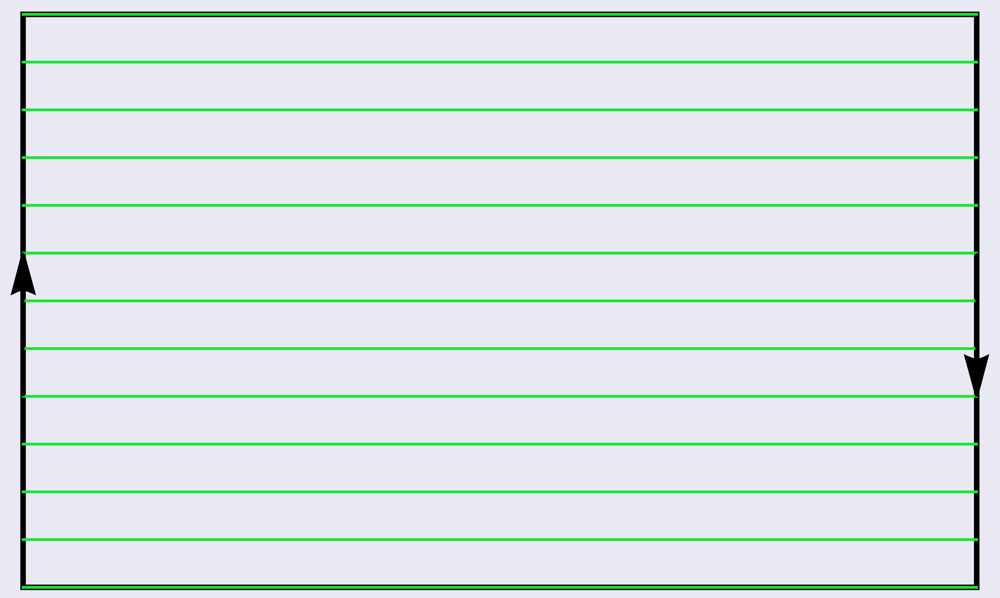
1. Möbius Band
2. Annulus
3. Torus
4. Klein Bottle

Definition (Foliation)

Let Σ be a compact, connected surface. A **foliation**, F of Σ is a decomposition of Σ into a union of disjoint, connected submanifolds of equal dimension. Each submanifold is called a **leaf** of the foliation.

A foliation is said to be **regular** if it has no singular points.





Foliation Types

There are two ways to construct foliations which are studied in this project- **suspensions** and **Reeb foliations**. It turns out that these two are fundamentally different up to homeomorphism.

Definition (Suspensions)

A **suspension** is a foliation made out of the foliation by horizontal lines on the unit square by using a function to identify the leaves of the trivial foliation with other leaves.

Explicitly, on the annulus $I \times S^1$ or the Möbius band $(I \times S^1)/\sim$ we make the assignment

$$(x, y) \mapsto (f^k(x), y + k) \text{ and}$$

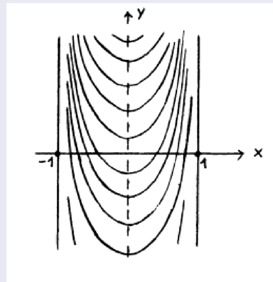
where $f^k(x)$ denotes composition of f k-times for some integer k. Abstractly, these are fiber bundles over S^1 with fibers moved by diffeomorphisms.

Reeb Foliaiton

Definition

Reeb Component In a similar fashion to the construction of suspensions, we can construct a **Reeb Component** by making the identification on the annulus or Möbius band

$$x \mapsto \frac{x^2}{1 - x^2} + y$$



Holonomy

Definition

The holonomy of a foliation encodes the behavior of the foliation around S^1 leaves. This is done by assigning to each path around the leaf a diffeomorphism of a transversal through the leaf. The set of germs of these diffeomorphisms forms a group called the holonomy group.

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For a representation $\chi : \pi_1(L, x) \to Diff([-1, 1], 0)$ we get the diagram

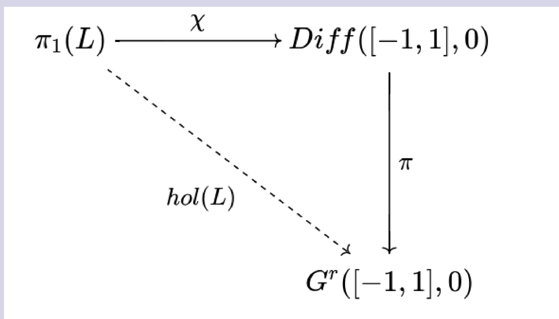
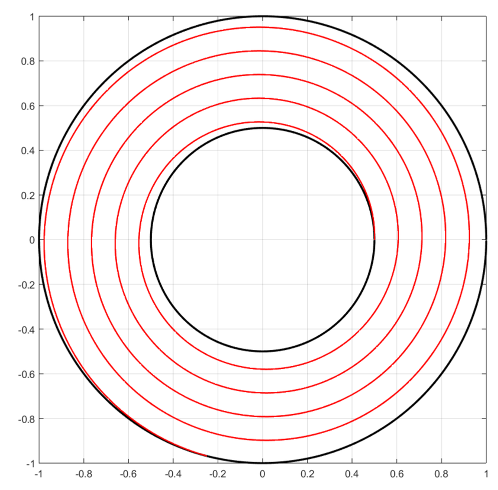


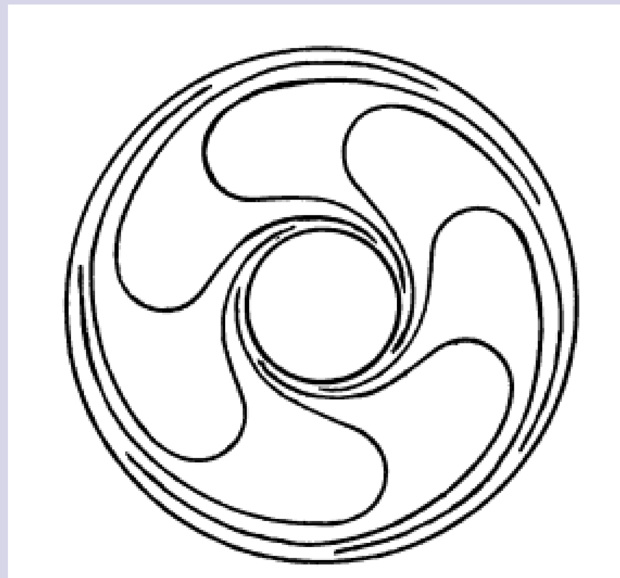
Figure: Caption

Graphics

Suspension on Annulus



Reeb Foliation on Annulus



Conclusions/Future Work

Using this information, we can classify any foliation on the annulus or Möbius band with a finite number of circle leaves.

Möbius Band:

1. As stated, with no interior circle leaves, we have the Reeb foliation
2. With any finite number of circle leaves, we can "cut" along them to obtain a decomposition into foliated annuli and Möbius bands with no interior circle leaves.

Annulus:

1. With the only circle leaves on the boundary, the holonomy is either "attracting" or "repelling" from these leaves
2. If the holonomy around both leaves is the same then we have the Reeb foliaiton
3. If they are opposite, it is a suspension
4. We "cut" along interior circle leaves to obtain smaller annuli with no interior circle leaves

Space of Foliations

Conjecture

The moduli space of foliations on the annulus with $n + 1$ circle leaves can be realized as the free group of words with n letters on the alphabet $S, \tilde{S}, R, \tilde{R}$ modulo the action of the mapping class group. For $G_{\mathbb{A}}$ the mapping class group, we write

$$F_n(\mathbb{A}) = W_n(S, \tilde{S}, R, \tilde{R})/G_{\mathbb{A}}$$

Relation to Holonomy

Given finite S^1 leaves, for each leaf, we have 2 pieces of holonomy data, excluding the exterior leaves which only have one. These can be thought of as "attracting" or "repelling." If we associate to each leaf its holonomy data we obtain

$$H_n(\mathbb{A}) = W_{2n}(a, r).$$

Each type of non-closed leaf can be associated to its holonomy data by

$$S \mapsto ar, \ \overline{S} \mapsto ra, \ R \mapsto aa, \ \overline{R} \mapsto rr.$$

This yields a bijective correspondence between the holonomy data and the space of foliations.

Acknowledgments

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References

Hector & Hirsch, Geometry of Foliations Part 1