Surfaces and Foliations

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Preliminaries: Surfaces & Euler Characteristic

Definition (Surfaces)

Let Σ be a two dimensional topological space covered by open sets U_α for some index α. If there exist homeomorphisms

$$\phi_\alpha : U_\alpha \to \mathbb{R}^2$$

for every α, then Σ is called a **surface**.

Definition (Euler Characteristic)

Let $\Omega \subset \mathbb{R}^2$ be a polyhedra. Then the **Euler Characteristic**, denoted $\chi(\Omega)$ is defined as

$$\chi(\Omega) = V - E + F$$

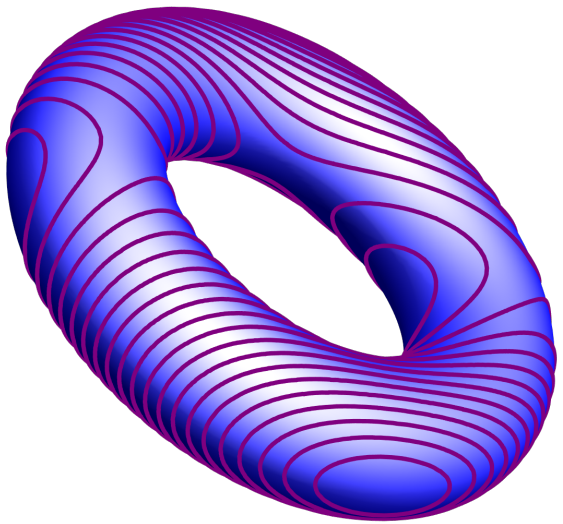
where V, E, F denote the vertices, edges, and faces, respectively, of Ω.

- We use the Euler Characteristic to determine which surfaces on which we focus our attention.

Foliations of Surfaces

Definition (Regular Foliation)

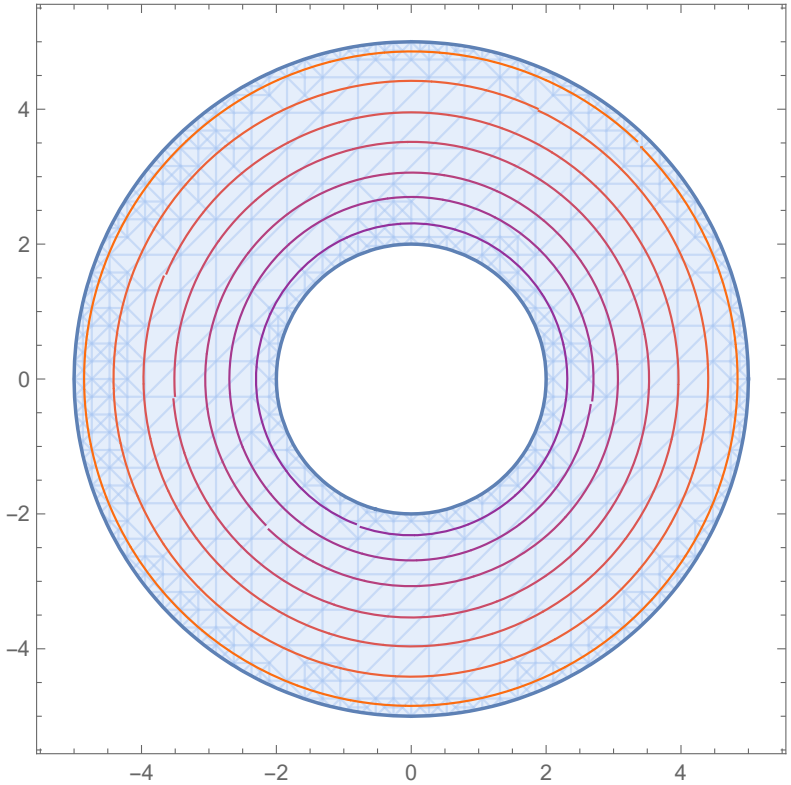
Let Σ be a compact, connected surface. A **foliation**, $\mathcal{F}$ of Σ is a decomposition of Σ into a union of disjoint, connected submanifolds of equal dimension. For our purposes, we focus on dimension one. Each submanifold is called a **leaf** of the foliation.

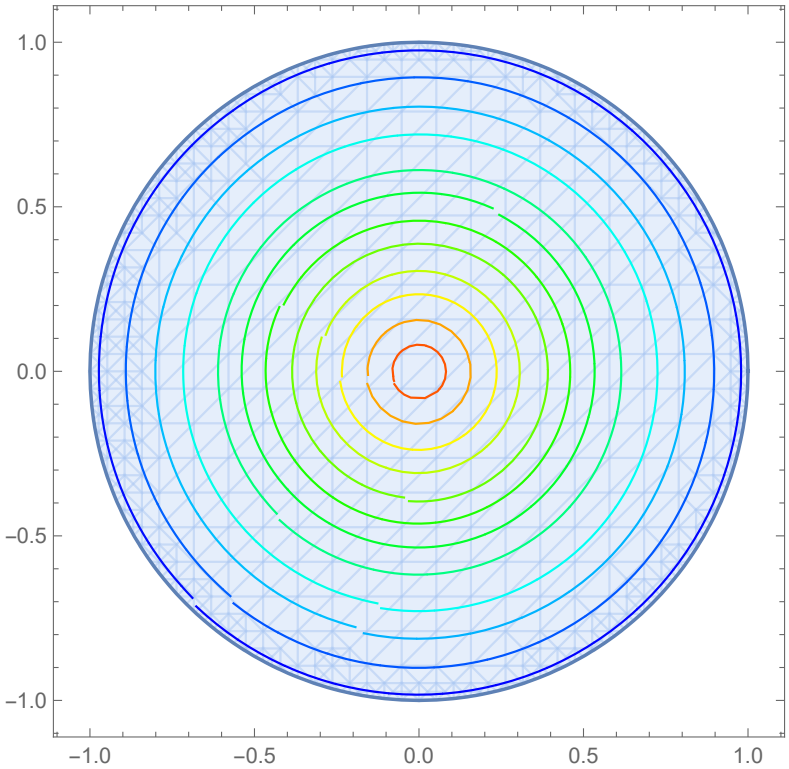


Poincaré-Kneser

Theorem (Poincaré-Kneser)

Let Σ be a compact, connected surface. Then Σ admits a regular foliation if and only if its Euler characteristic is zero.





Constructing Foliations

Definition (Suspensions)

A **suspension** is a foliation made out of the foliation by horizontal lines on the unit square by using a function to identify the leaves of the trivial foliation with other leaves.

Explicitly, to find the suspension on the annulus $\mathbb{A}$ and the Möbius band $\mathbb{M}$, we make the identification on the unit square:

$$(x, y) \sim (x + 1, f(y))$$

where $f : [0, 1] \to [0, 1]$ is a diffeomorphism.

Suspensions cont...

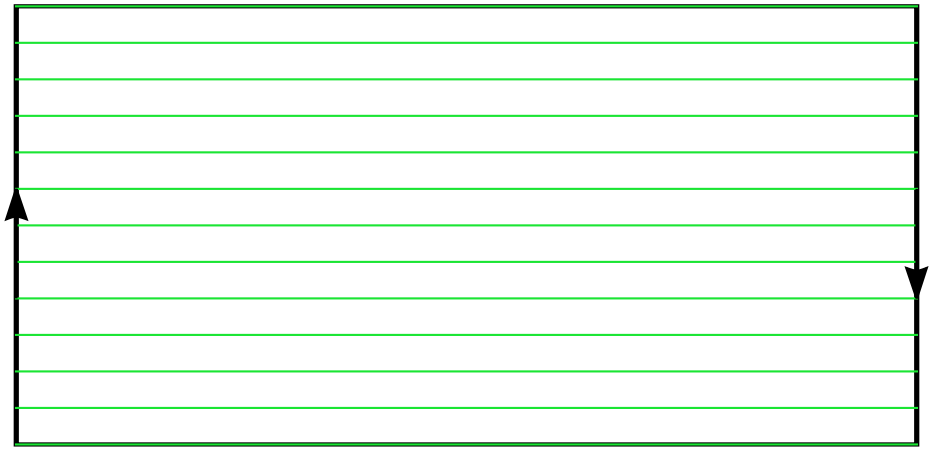


Figure: Trivial foliation on Möbius band, already been suspended.

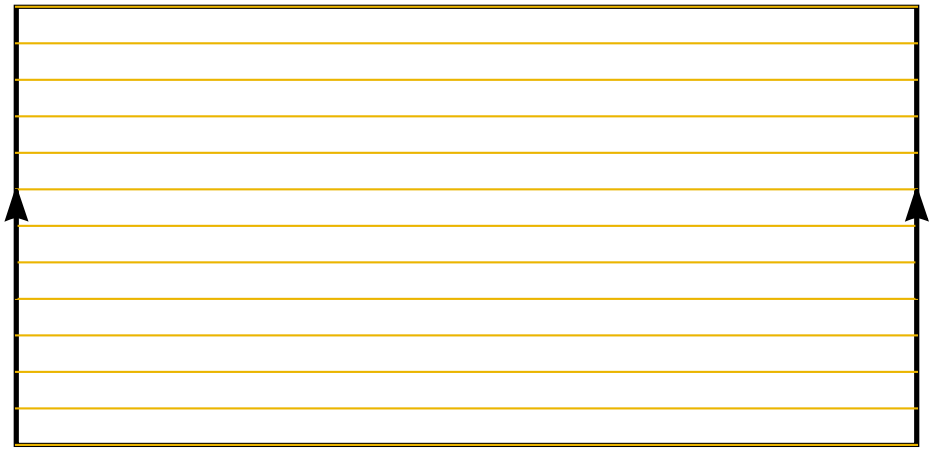
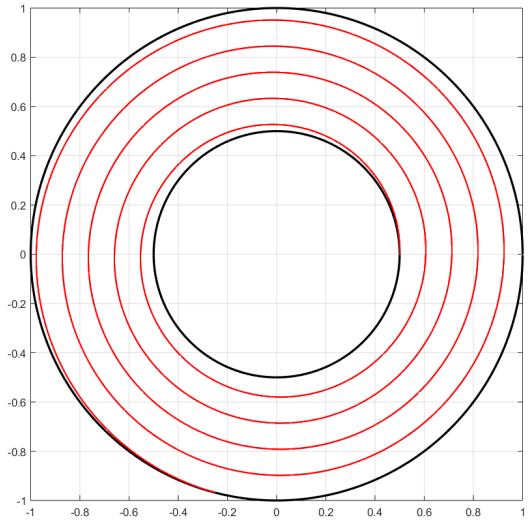
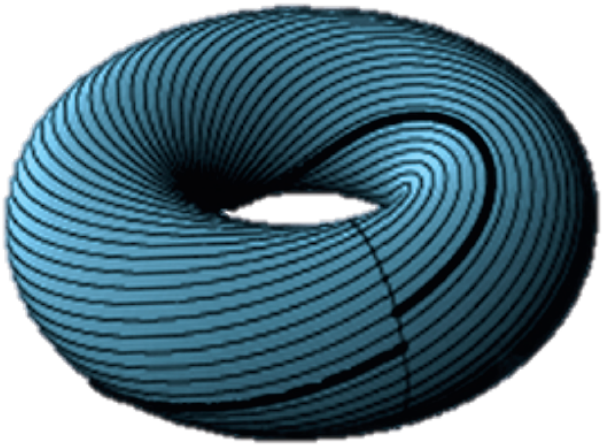


Figure: Annulus, in suspense

Suspension on the Annulus and Torus





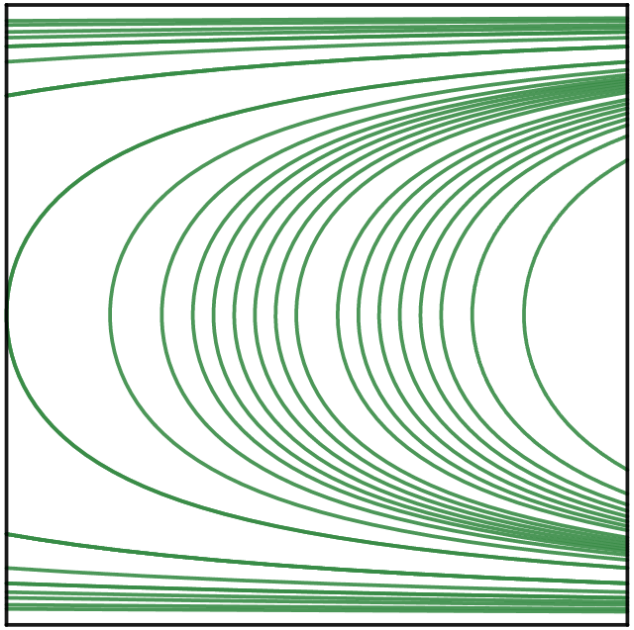
Reeb Foliation

Definition (Reeb Component)

In a similar fashion to the construction of suspensions, we can construct a **Reeb Component** by making the identification on the annulus or Möbius band

$$y \mapsto \frac{y^2}{1-y^2} + x$$

where the foliation is the family of curves for $x \in \mathbb{R}$.

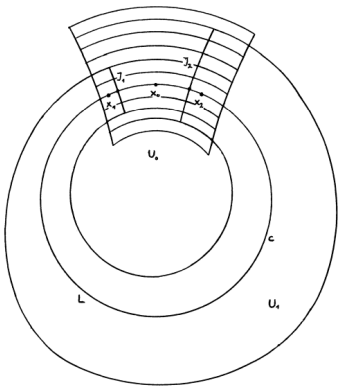


Abstract Nonsense: Holonomy

Definition

A **circle leaf** is leaf of a foliation which is homeomorphic to a circle. For a foliated surface $(\Sigma, \mathcal{F})$, the **holonomy** of a leaf $L \in \mathcal{F}$ at a point $x \in L$ encodes the behavior of the other leaves around L. This behavior is encoded in a group called the **holonomy group** of L , written $hol(L)$.

While this definition is somewhat abstract, it turns out that the holonomy of a foliation provides a wealth of information about the foliation.



Classification

Based on the holonomy and basic facts about foliations, we obtain the following facts

- The Reeb foliation cannot be constructed from a suspension as suspensions are defined here.
- Any two suspensions constructed on the Annulus with no interior circle leaves are homeomorphic.
- A foliation with a finite number of circle leaves is built by gluing a finite number of Reeb components and suspensions.
- A foliation on the Möbius band with no interior circle leaves must be the Reeb foliation.
- Every foliation on the Klein Bottle has at least one circle leaf.

Classification on Möbius

Using this information, we can classify any foliation on the annulus or Möbius band with a finite number of circle leaves.

Möbius Band:

- As stated, with no interior circle leaves, we have the Reeb foliation
- With any finite number of circle leaves, we can ”cut” along them to obtain a decomposition into foliated annuli and Möbius bands with no interior circle leaves.

Classification on Annulus

Annulus:

- With the only circle leaves on the boundary, the holonomy is either ”attracting” or ”repelling” from these leaves
- If the holonomy around both leaves is the same then we have the Reeb foliation
- If they are opposite, it is a suspension
- We ”cut” along interior circle leaves to obtain smaller annuli with no interior circle leaves

Space of Foliations on Annulus

Definition

A **moduli space** is a topological space whose points correspond to other spaces. In studying the foliations we can have on a given surface, it is natural to ask about the properties of the space of all such foliations.

Conjecture

The moduli space of foliations on the annulus with $n+1$ circle leaves can be realized as the free group of words with n letters on the alphabet $S, \tilde{S}, R, \tilde{R}$ modulo the action of the mapping class group. For $G_{\mathbb{A}}$ the mapping class group, we write

$$\mathcal{F}_n(\mathbb{A}) = W_n(S, \tilde{S}, R, \tilde{R})/G_{\mathbb{A}}$$

The alphabet represents the possible leaves up to homeomorphism. There are two possible suspensions and two possible Reeb components between each of the circle leaves.

Further Research

- Study topology of space of foliations
- Look for space of foliations with ”nice” topology
- Classify foliations based on holonomy data- have already constructed bijection to space of annuli together with holonomy data

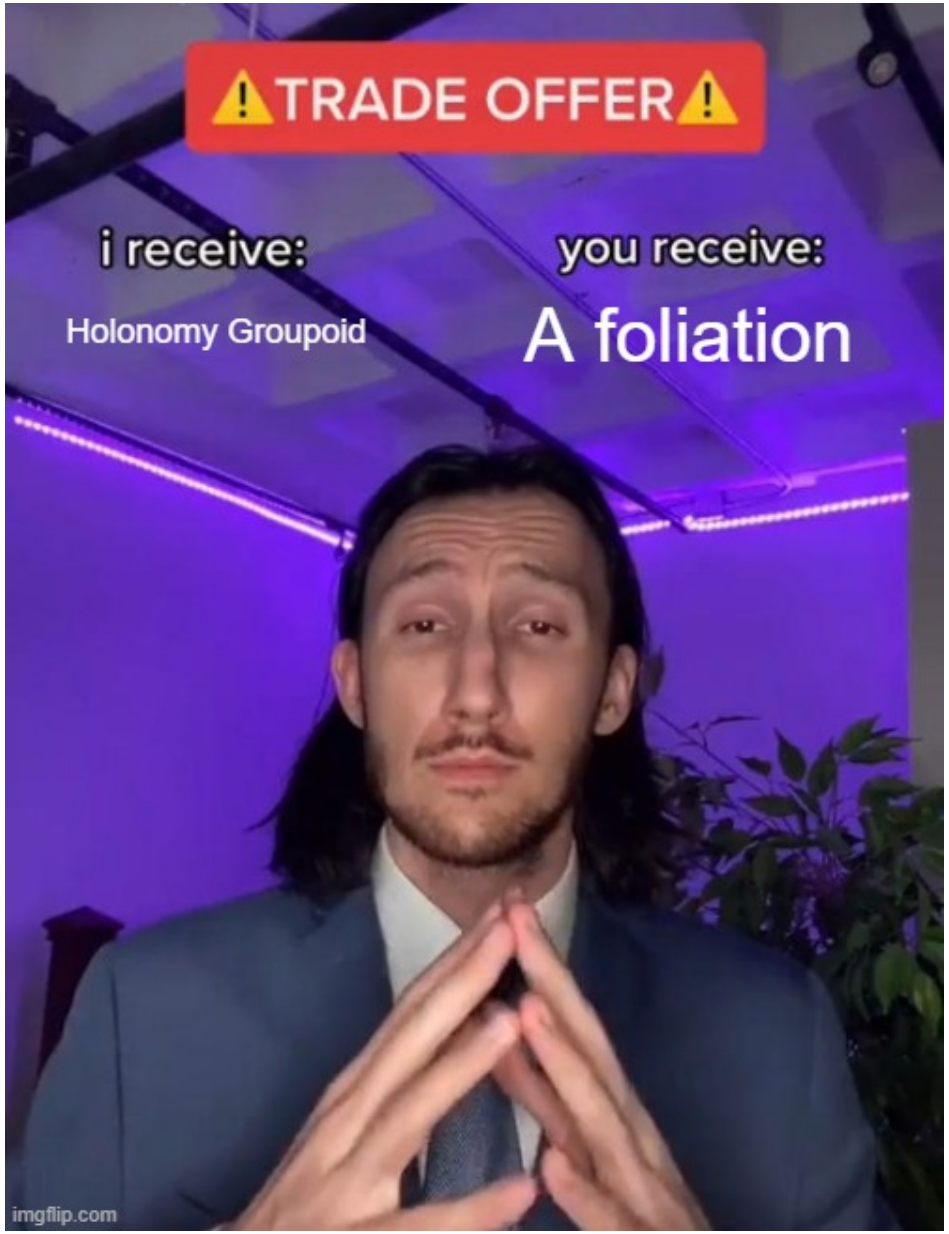


Figure: Future Research

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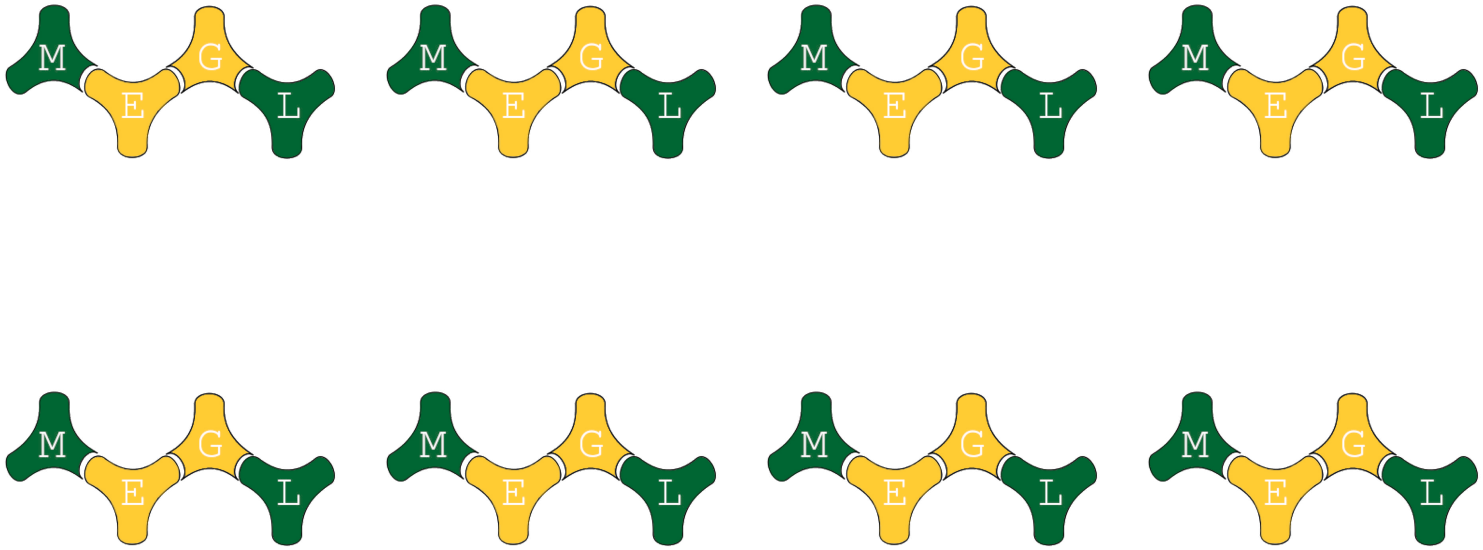

Sources

Hector & Hirsch, Geometry of Foliations Part 1
Zeeman, Classification of Surfaces