Resolving Neural Ideals

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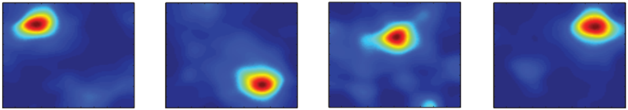
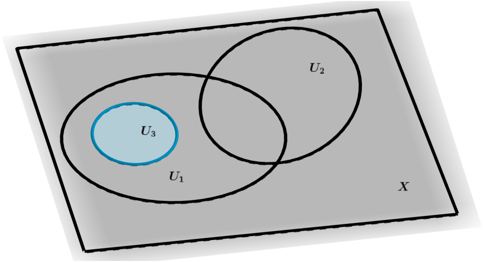


Heat map of mice place cell activity in an experiment [3]

- **Place Cells** - Class of neuron which "process" the location of an animal in some region of space.
- **Stimulus Space** - Closed region X containing possible stimuli.
- **Place Field** - Area corresponding to a place cell which fires whenever the animal enters.

Receptive Field Structure

- **Receptive Field (RF)** - Set $U_i \subset X$ corresponding to some place field of the i^{th} neuron in a population of n.
- **RF code** - Set of all binary codewords corresponding to stimuli in X. An entry in codeword $c \in \mathcal{C} \subseteq \{0,1\}^n$ takes a value of 1 when stimulated.



Receptive field of three neurons with an activation corresponding to a codeword of $c = (1, 0, 1)$.

Pseudo-monomial Ideals

- **Pseudo-monomials** are of the form

$$\prod_{i\in\sigma} x_i \prod_{j\in\tau}(1-x_j)$$

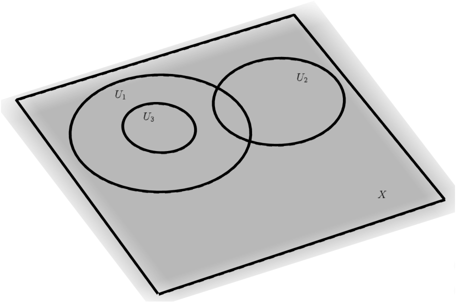
  where $\sigma \cap \tau = \emptyset$ and $\sigma, \tau \subseteq \{1, ..., n\}$ corresponded to indices of specific codewords in $\mathcal{C}$.

- **Pseudo-monomial Ideals** are sets of sums of pseudo-monomials

- The key observation is that there exists a correspondence between the RF structure and pseudo-monomial ideals

Example

- Let J be an ideal in $\mathbb{F}_2[x_1, x_2, x_3]$ corresponding to the previous RF structure. Then,



$$U_1 \cap U_2 \cap U_3 = \emptyset \implies x_1 x_2 x_3 \in J$$

$$U_3 \subseteq U_1 \implies x_3(1 - x_1) \in J$$

$$U_2 \cap U_3 = \emptyset \implies x_2 x_3 \in J$$

Hence $J = (x_1 x_2 x_3, x_3(1 - x_1), x_2 x_3)$.

Polarizing Pseudo-monomials [4]

- Pseudo-monomial ideals are difficult to work with so we use a method called **polarization** to write them in a “nicer” form, allowing us to use familiar algebraic methods

- Using the previous example, the relation is as follows:

  - Let J be a pseudo-monomial ideal in $\mathbb{F}_2[x_1, x_2, x_3]$ given by

$$J = (x_1x_2x_3, x_3(1 - x_1), x_2x_3)$$

  - Now consider $\mathcal{P}(J)$ to be in the polarized ring $\mathbb{F}_2[x_1, x_2, x_3, y_1, y_2, y_3]$ and make the identification

$$x_i(1 - x_j) \to x_iy_j$$

  - Hence we obtain

$$\mathcal{P}(J) = (x_1x_2x_3, x_3y_1, x_2x_3)$$

Canonical Form of a Neural Ideal

- **Canonical Form of Pseudo-monomial Ideal** - Encodes minimal relations of RF structure in a way which can be directly read off from given ideal

- CF of neural ideal is uniquely tied to RF structure- two ideals are the same if and only if they have the same canonical form

- **Three Types of Relations:**

$$\textbf{1.)}\ \ x_\sigma \iff \bigcap_{i\in\sigma} U_i = \emptyset$$

$$\textbf{2.)}\ \ \prod_{i\in\sigma} x_i \prod_{j\in\tau}(1-x_j) \iff \bigcap_{i\in\sigma} U_i \subseteq \bigcup_{j\in\tau} U_j$$

$$\textbf{3.)}\ \ \prod_{j\in\tau}(1-x_j) \iff X \subseteq \bigcup_{j\in\tau} U_j$$

Canonical Form of a Neural Ideal Example

- Computing the canonical form involves eliminating shared indices between generators
- The canonical form allows one to identify obstructions to convexity, an integral property in studying a set's geometric and topological structure
- Using our trusty example,

$$J = (x_1x_2x_3, x_3y_1, x_2x_3)$$

- We add generators for shared indices and remove generators which are divisible by others

$$CF(J) = (x_3y_1, x_2x_3)$$

Free Resolutions and Betti sequences

From the relations in CF(J), we can realize a **free resolution** given by the sequence

$$\mathcal{F}\colon 0 \longrightarrow R^{\beta_d} \xrightarrow{\partial_d} R^{\beta_{d-1}} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R \longrightarrow R/J \longrightarrow 0$$

where $R = \mathbb{F}_2[x_1, \ldots, x_n]$ and $1, \beta_1, \ldots, \beta_d$ is a **Betti sequence** or sequence of **Betti numbers**.

- We suspect that certain characteristics of Betti sequences may be invariant among classes of canonical forms, thus allowing us to meaningfully classify RF structures.

Free Resolutions and Betti sequences - example

Example

(Theorem 7.1[1]; length 2 example) The canonical form of a polarized neural ideal in $K = R[y_1, \ldots, y_7]$ given by

$$\langle x_1 g_1, x_2 y_1 g_2, y_2, x_2[g_1 g_2], [g_1 g_2 g_3], y_1[g_2 g_3] \rangle$$

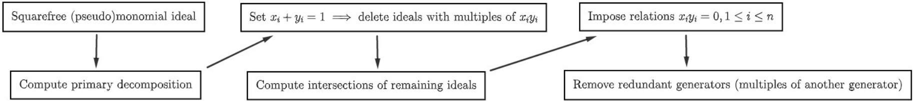
where $g_1 = x_4 x_5 x_6$, $g_2 = x_4 x_7$, $g_3 = x_4 x_6 x_7$ and $[g_p, g_q]$ denotes the LCM of g_p and g_q with the g_k's indicating specification on a family of canonical forms of this type. This yields the resolution

$$0 \longrightarrow K \longrightarrow K^5 \longrightarrow K^5 \longrightarrow K \longrightarrow 0$$

which has the Betti sequence 1,5,5,1.

- Notice that the above example's Betti sequence is symmetric; this is one of many characteristics we tried to match up with classes of canonical forms as found in Section 7 of [1].

Computing Canonical Form of a Neural Ideal



- Computing the canonical form can quickly become time consuming
- We created a script in Macaulay2 to automate finding the canonical form using Algorithm 3.2 from [1].

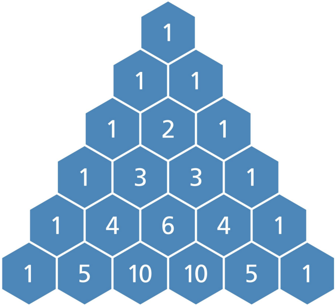
```
{x y , x x y , x y }
  3 5   1 3 4   4 5

i9 : res I

       1       3       2
o9 = R  <-- R  <-- R  <-- 0

     0       1       2       3
```

- Once we have the canonical form, we compute the resolution of the ideals.
- From these resolutions, we look at patterns in the Betti numbers.



- For example, when the Betti numbers correspond to Pascal's triangle, we know we have a Taylor resolution.

Results

- We found that there are Betti sequences that are symmetric but not Taylor.
- In other cases, we found some conditions under which the Betti numbers change based off of the addition of generators which do not change our canonical form
- We also found classes of ideals whose Betti sequence is identical regardless of our choice of "g"s. Moving forward we would like to investigate if there is some algebraic invariant that links these cases.
- We also plan to compile our code into a Macaulay2 package.

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References

[1] Geller Hugh, Rebecca R.G., Canonical forms of neural ideals, https://doi.org/10.48550/arXiv.2209.09948

[2] Curto Carina, Itskov Vladimir, Veliz-Cuba Alan, Youngs Nora, The neural ring: An algebraic tool for analyzing the instrinsic structure of neural codes, https://doi.org/10.48550/arXiv.1212.4201

[3] C. Giusti, E. Pastalkova, C. Curto, and V. Itskov. Clique topology reveals intrinsic geometric structure in neural correlations. Proc. Natl. Acad. Sci., 112(44):13455–13460, 2015.

[4] S. Gunturkun, J. Jeffries, J. Sun, Polarization of Neural Rings, https://arxiv.org/abs/1706.08559