Dynamics on Polynomials over Finite Fields

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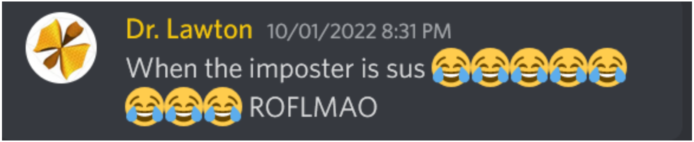


Definition

- A group is a set with an associative binary operation with an identity and per-element inverses.
- A group action of a group G on a set X is an operation $\cdot : G \times X \to X$ such that
  1. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, and
  2. $e \cdot x = x$,

  for all $g_1, g_2 \in G$ and $x \in X$, and e is the identity of G.
- $\mathrm{GL}_m(\mathbb{F}) = \{A \in \mathrm{Mat}_{m\times m} \mid \det(A) \neq 0\}$ is the group of invertible matrices over a field $\mathbb{F}$.
- $\mathbb{F}[x_1, \ldots, x_m]$ is the set of polynomials with coefficients in $\mathbb{F}$ of the formal symbols $x_1, \ldots, x_m$.

Definitions and Notation

Definition

Let $m, d \in \mathbb{N}^+$ and $\mathbb{F}$ be a finite field.
We define $\mathcal{P}^*_{m,d}(\mathbb{F}) = \{f \in \mathbb{F}[x_1, \dots, x_m] \mid \mathrm{totaldeg}(f) \le d\} - \{0\}$, and let $\mathrm{GL}_m(\mathbb{F}) \circlearrowleft \mathcal{P}^*_{m,d}(\mathbb{F})$ by

$$g \cdot f(\vec{x}) = f(g^{-1}\vec{x}),$$

for $g \in \mathrm{GL}_m(\mathbb{F}), f \in \mathcal{P}^*_{m,d}(\mathbb{F})$, and formal symbols $\vec{x} = (x_1, \dots, x_m)$. This action is linear and degree-preserving.

Note

In cases where $\mathbb{F} = \mathbb{Z}_p$ for some prime p, we amend our notation to $\mathcal{P}^*_{m,d,p} := \mathcal{P}^*_{m,d}(\mathbb{F})$, and we mean that $\mathbb{F} = \mathbb{Z}_p$ when we say "let p be ..."

Remark

$|\mathcal{P}^*_{m,d}(\mathbb{F})| = |\mathbb{F}|^{\binom{m+d}{m}} - 1$ (Proof: "Stars and Bars").

Simple Example

Example 1.1

Let $m = d = p = 2$. This is the effect of a generator of $\mathrm{GL}_2(\mathbb{Z}_2)$ on $x^2 + xy$. First compute multiplication on the formal symbols,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y \end{pmatrix},$$

and then compose with the polynomial function:

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} (x^2 + xy) &= (x+y)^2 + (x+y)y \\ &= x^2 + 2xy + y^2 + xy + y^2 \\ &= x^2 + 2xy + xy + 2y^2 \\ &= x^2 + xy. \end{aligned}$$

Note: $2xy = 2y^2 = 0$ since $2 = 0 \bmod 2$.

Simple Example Cont.

Definition

For a polynomial $f \in \mathcal{P}^*_{m,d}(\mathbb{F})$, its orbit is $\mathrm{Orb}(f) = \{g \cdot f \mid g \in \mathrm{GL}_m(\mathbb{F})\}$, and its stabilizer is $\mathrm{Stab}(f) = \{g \in \mathrm{GL}_m(\mathbb{F}) \mid g \cdot f = f\}$. An element f of $\mathcal{P}^*_{m,d}(\mathbb{F})$ is *fixed* if $\mathrm{Orb}(f) = \{f\}$, or equivalently, $\mathrm{Stab}(f) = \mathrm{GL}_m(\mathbb{F})$. All constant polynomials will be fixed.

Example 1.2

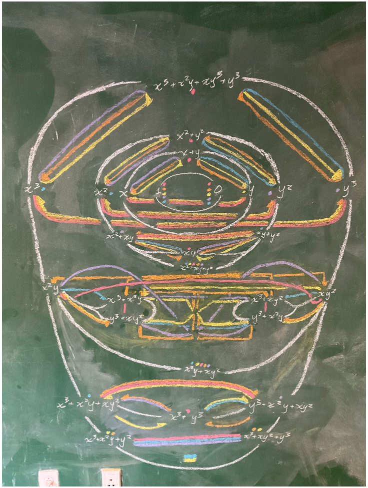
In the previous case,

$$\mathrm{Orb}(x^2 + xy) = \{x^2 + xy, y^2 + xy, xy\}$$

and

$$\mathrm{Stab}(x^2 + xy) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

Orbits when $m = 3$ and $d = p = 2$:



Linear Polynomials

Definition

A polynomial is *homogeneous* if each term is of the same total degree. We define
$\mathcal{H}_{m,d}(\mathbb{F}) = \{f \in \mathcal{P}^*_{m,d}(\mathbb{F}) \mid \text{totaldeg}(f) = d, \text{ and } f \text{ is homogeneous}\}$. The action preserves homogeneity.

Theorem

Let $d = 1$, $m \in \mathbb{N}^+$, and $\mathbb{F}$ be a finite field. Define $\mathcal{M}_{m,d}(\mathbb{F}) = \{\text{Orb}(f) \mid f \in \mathcal{P}^*_{m,d}(\mathbb{F})\}$. We get that $|\mathcal{M}_{m,d}(\mathbb{F})| = 2|\mathbb{F}| - 1$, and if $f = q + k$, where $q \in \mathcal{H}_{m,d}(\mathbb{F})$ and $k \in \mathbb{F}$,

$$\text{Orb}(f) = \text{Orb}(q) + \text{Orb}(k) = \begin{cases} \mathcal{H}_{m,d}(\mathbb{F}) + k & \textit{if } q \neq 0 \\ \{k\} & \textit{otherwise} \end{cases}$$

Proof and Corollary

Lemma

Let $q \in \mathcal{H}_{m,1}(\mathbb{F})$. *There exists* $g \in \mathrm{GL}_m(\mathbb{F})$ *such that* $q = gx_1$.

Proof (Sketch).

By linearity, we get that if f decomposes into $q + k$, where $q \in \mathcal{H}_{m,1}$ and $k \in \mathbb{F}$, $\mathrm{Orb}(f) = \mathrm{Orb}(q) + k$. It suffices to show that $\mathrm{Orb}(q) = \mathcal{H}_{m,1}(\mathbb{F})$. Let $q_1, q_2 \in \mathcal{H}_{m,1}(\mathbb{F})$ and define g_1 and g_2 by the previous lemma. Then $q_2 = g_2 g_1^{-1} q_1$, so $\mathrm{Orb}(q) = \mathcal{H}_{m,1}(\mathbb{F})$. □

Corollary

All degree one polynomials have an orbit of size $|\mathbb{F}|^m - 1$, *and therefore have a stabilizer of order* $\prod_{i=1}^{m-1} \left(|\mathbb{F}|^m - |\mathbb{F}|^i\right)$ *via the Orbit-Stabilizer Theorem.*

Trying to work out $m = p = 2$, $d \in \mathbb{N}^+$

Theorem

Let $m = p = 2$ and $2 \leq d \in \mathbb{N}^+$. There exists polynomials that are fixed of total degree d, precisely, linear combinations of

$$\begin{cases} x^d + x^{\frac{d}{2}} y^{\frac{d}{2}} + y^d & \text{if } d \text{ is even} \\ x^{d-1}y + xy^{d-1} & \text{if } d \text{ is odd} \\ x^s y^r + x^r y^s & \text{if } d = s + r, \text{ where } s \text{ and } r \text{ are powers of } 2 \end{cases}$$

We do not know if these are the only fixed polynomials in this case.

Remark

Other than those above, we have yet to find any non-constant fixed points.

Super Ziqi Conjecture

Excluding constants, there are *NO* fixed points if $d \neq 2$ or $p \neq 2$.

The Data We Have (Initial Naive Implementation)

Fixed Point Data

$m = 2$

d \ p	2	3	5	7
1	$\star$	$\star$	$\star$	$\star$
2	$\square$	$\star$	$\star$	$\star$
3	$\square$	—	—	—
4	$\square$	—	—	—

$m = 3$

d \ p	2	3	5
1	$\star$	$\star$	$\star$
2	$\star$	—	—

Legend:

$\star$: No non-constant fixed points

$\square$: Existence of non-constant fixed points

—: No data

Note: Cardinalities blow up quickly. $|\mathcal{P}^*_{5,3,5}| \approx 1.39 \times 10^{39}$.

COUNTEREXAMPLE TO SUPER ZIQI CONJECTURE

A random paper that we found 2 weeks ago from 1911 shows us that this polynomial in particular is fixed if $m = 2$ and $p = 3$!!

$$x^6 + x^4y^2 + x^2y^4 + y^6$$

Wow. Look at that. So impressive, so cool. It doesn't care about what matrices think about it.

Asymptotic Transitivity

Definition

The action is *transitive* if $\mathcal{M}_{m,d}(\mathbb{F}) = \{\mathcal{P}^*_{m,d}(\mathbb{F})\}$, or that for every pair of f_1 and f_2, there exists $g \in \mathrm{GL}_m(\mathbb{F})$ such that $g \cdot f_1 = f_2$.
Let $M = \max\{|O| \mid O \in \mathcal{M}_{m,d,p}\}$. We say that the action is *asymptotically transitive* if

$$\lim_{p\to\infty} \frac{M}{|\mathcal{P}^*_{m,d,p}|} = 1.$$

The action is not transitive. However, is it ever asymptotically transitive for specific choices of m and d?

Remark

If $m^2 \leq \binom{m+d}{m}$, $\lim_{p\to\infty} |\mathrm{GL}_m(\mathbb{Z}_p)|/|\mathcal{P}^*_{m,d,p}| = 0$, and we have no hope of asymptotic transitivity.

Additional Theorems

Additional things we have proved:

Theorem (Fixed Points)

Every fixed polynomial must be symmetric, and $|\mathbb{F}| - 1$ *must divide each component of the multiexponent for each term. This is not a sufficient condition.*

Theorem (Maximal Orbits)

An orbit is _maximal_ *if it has the same cardinality as* $\mathrm{GL}_m(\mathbb{F})$*. There will always be a maximal orbit if* $d > m$*. In the special case where* $p = 2$*, there will always be a maximal orbit if* $d \geq m$*.*

Review of Accomplishments

- Came up with the problem for the project.
- Developed code to compute smaller cases that helped us prove some theorems.
- Proved that the action is linear, preserves homogeneity, and preserves multiplication.
- Solved the cases for $m = 1$ and $d = 1$ fully.
- Finding classes of fixed points for $m = p = 2$.
- Found necessary conditions for maximal orbits and fixed points.
- (Mostly) developed the improved code that will be use next semester.
- Basic observations for asymptotic transitivity using cardinality arguments.

Next Semester

- Interface our current results with the “local dynamics” of $\mathrm{Stab}(f) \circlearrowleft V(f)$, the variety of f.
- Finish and run the faster and more efficient code to find more patterns.
- Read *On Dickson’s theorem on invariants* by Robert Steinberg, which we think solves the fixed point problem and in particular disproves the Super Ziqi Conjecture.
- Work more on asymptotic transitivity, and find when it occurs.
- Sharpen the bound for maximal orbits.
- Wrap up the $d = 2$ case in general, and start working on $d = 3$.

Thanks

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