Geometric Desingularization

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Inspiration for Research

- Bramburger & Henderson Research in FKPP-Burgers system [1]
  - Coupled reaction-advection-diffusion system with a large parameter ρ
- Traveling wavefront solutions only exist for *some* wave speeeds $\tilde{c}$
  - $(T, u) = (T(x - \tilde{c}t), U(x - \tilde{c}t))$
- For a given ρ, the set of admissible speeds is $[c_*(\rho), \infty)$
- Theorem: $(\frac{3}{2})^{1/3} \leq \liminf_{\rho\to\infty} \frac{c_*(\rho)}{\rho^{1/3}} \leq \limsup_{\rho\to\infty} \frac{c_*(\rho)}{\rho^{1/3}} \leq \sqrt{3}$ (Bramburger & Henderson 2021)
- Theorem: $\lim_{\rho\to\infty} \frac{c_*(\rho)}{\rho^{1/3}} = \sqrt[3]{\frac{3}{2}}$ (HKMTW)

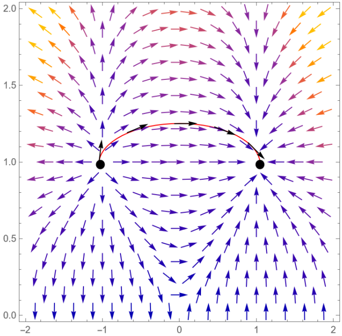
ODE's, Re-scaling and Desingularization

- The system of ODE's derived by Bramburger & Henderson
  - $\dot{T} = -\tilde{c}T + UT + \frac{1}{2\rho}U(2\tilde{c} - U)$
  - $\dot{U} = \frac{\rho}{U - \tilde{c}}T(1 - T)$
- We'll use the following re-scaling, $\varepsilon^3 = \frac{1}{\rho}$, $c = \frac{\tilde{c}}{\varepsilon}$, and $W = U\varepsilon$ to change coordinates and rescale time
  - $T' = \varepsilon\dot{T}$
  - $T' = -cT + WT + \frac{1}{2}W\varepsilon^2(2c - W)$
  - $W' = \frac{1}{W - c}T(1 - T)$

Rescale, Reduce, Desingularize

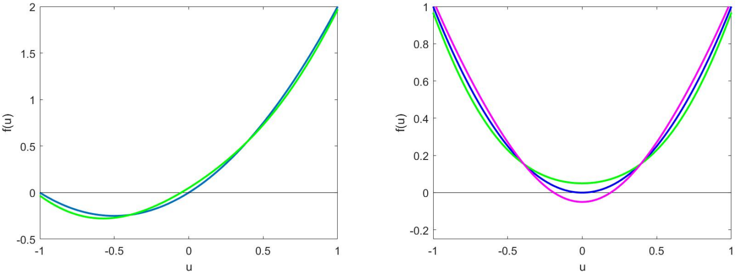
- Singularity at $W = c$
- Multiply by $c - W$ to desingularize
  - $T' = -T(c - W)^2 + \frac{1}{2}\varepsilon^2 W(2c - W)(c - W)$
  - $W' = -T(1 - T)$
- Consider only the $\varepsilon = 0$ case
  - $T' = -T(c - W)^2$
  - $W' = -T(1 - T)$
- Fixed points at $(0, 0)$ and $(1, c)$ found by inspection

Heteroclinic Orbits



- When working with differential equations, where is the rate of change zero?
- Solution connecting an unstable fixed point and a stable fixed point.

Hyperbolicity



- With hyperbolic fixed points, reduced linear dynamics <u>**will**</u> approximate non-linear dynamics

- With non-hyperbolic fixed points, reduced linear dynamics <u>**may not**</u> approximate non-linear dynamics

- (See Hartman-Grobman Theorem for more details)

Relating T and W ($\varepsilon = 0$)

- A separable ODE is found by relating the two derivatives(" $'$ " being $\frac{d}{d\tau}$)
  - $\frac{dW}{dT} = \frac{dW/d\tau}{dT/d\tau} = \frac{1-T}{(W-c)^2}$
- Solving this using separation of variables, an explicit function of $W(T)$ is found
  - $W(T) = c + \sqrt[3]{\frac{3}{2}(1-T)^2 + K}$

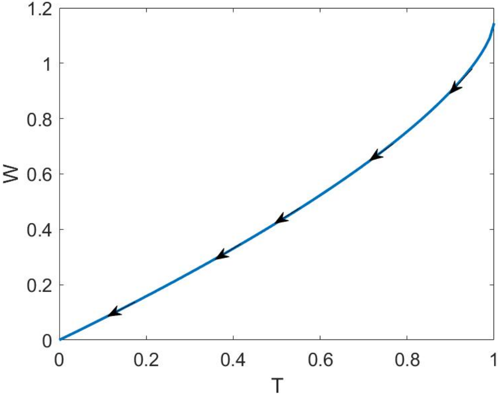
Solving for c

- From the separation of variables step, we obtained the equation
  - $\frac{1}{3}(W - c)^3 = -\frac{1}{2}(1 - T)^2 + K$, where K is the integration constant.
- If a heteroclinic orbit exists, use the fixed points $(0, 0)$ and $(1, c)$ as boundary conditions

Solving for c

- Using the point (1,c), find K
  - $\frac{1}{3}(c-c)^3 = -\frac{1}{2}(1-1)^2 + K$
  - $K = 0$
- Using the point $(0,0)$, find c
  - $\frac{1}{3}(0-c)^3 = -\frac{1}{2}(1-0)^2 + K$
  - $c = \sqrt[3]{\frac{3}{2}}$
- So our theorem is true!
  - ...if we assume a heteroclinic orbit exists...

Graphic Illustration of Heteroclinic orbit connecting to non-hyperbolic fixed point at c



The fixed point, $(1, c)$ is non-hyperbolic, so more advanced techniques will be needed to recover hyperbolicity

Implicit Function Theorem

- How do we know if a heteroclinic orbit exists?
    - There will be solutions passing through *one* fixed point, but what about two?
- Implicit Function Theorem! An implicit function exists if:
    - Can construct some $\Phi(c, \varepsilon)$ with $\Phi(c^*, 0) = 0$ along some section $\Sigma = \{(T, W) : T = \frac{1}{2}\}$
    - The parameters c, ε are smooth along these sections
    - $\frac{\partial \Phi}{\partial c}(c^*, 0) \neq 0$ along Σ

Defining Φ

- Let h_s be a solution through the stable fixed point
- Let h_u be a solution through the unstable fixed point
- We have a heteroclinic orbit when h_s is h_u, so let

$$\Phi(c, \varepsilon) = h_s(c, \varepsilon) - h_u(c, \varepsilon)$$

- So we have a heteroclinic orbit if we can show that this Φ is zero.
  - Easy to show $\Phi = 0$ when fixed points are hyperbolic
  - But (1,c) is non-hyperbolic, so more advanced techniques are needed

'Blow-up' at Non-Hyperbolic Fixed Point (1,c)

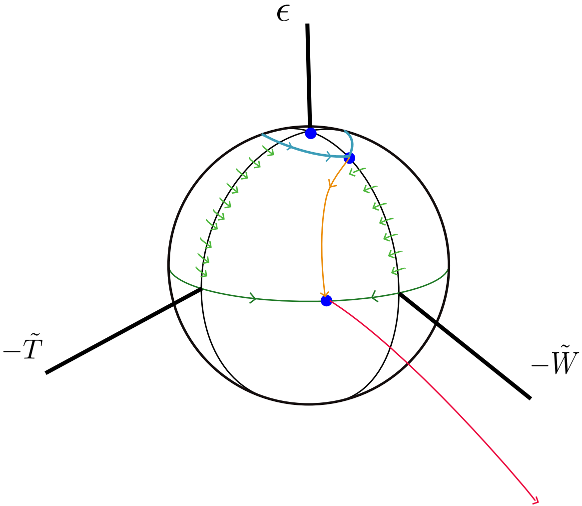
'Blowing-up' the (1,c) fixed point into an ellipsoid projected onto three planes to analyze. First, we must re-center the origin via the following translation: $\tilde{W} = W - c$ and $\tilde{T} = T - 1$

Resulting equations after translation:

$$\tilde{T}' = -\tilde{W}^2(\tilde{T} + 1) - \frac{1}{2}\tilde{W}\varepsilon^2(c^2 - \tilde{W}^2)$$
$$\tilde{W}' = \tilde{T}(\tilde{T} + 1)$$

Once the origin is translated, the coordinates $(\tilde{T}, \tilde{W}, \varepsilon)$ will be mapped via a quasi-homogeneous blow up to (r, θ, φ) in the following way:
$\tilde{T} = r^2 cos\theta sin\varphi$, $\tilde{W} = rsin\theta sin\varphi$, and $\varepsilon = cos\varphi$

Geometric Picture of Quasi-Homogeneous Blow-up



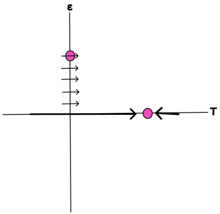
Analyzing Dynamics on the Ellipsoid

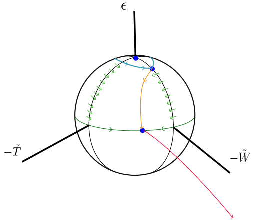
- Three fixed points vs one
- Projective charts instead of spherical coordinates for easier analysis
- Trapping region to ensure connection

In the following section, we show the analysis of the charts and construction of the trapping region

Chart k_W

Change of coordinates: $\varepsilon = \varepsilon_1 r_1$, $\tilde{W} = -r_2$, $\tilde{T} = T_1 r_1^3$

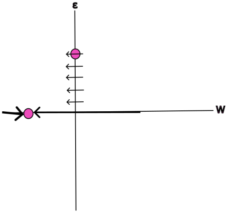


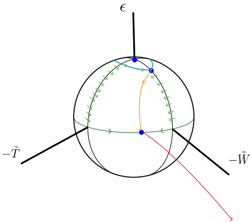


The equations in this chart:

$$T_1' = \frac{3}{2}T_1 + 1 - \frac{\varepsilon_1 c^2}{2}$$
$$\varepsilon_1' = \frac{\varepsilon_2 T_1}{2}$$

Change of Coordinates: $\varepsilon = \varepsilon_2 r_2$, $\tilde{W} = r_2^2 W_2$, $\tilde{T} = r_2^3$

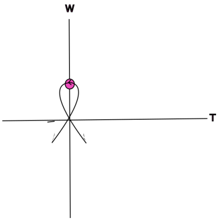


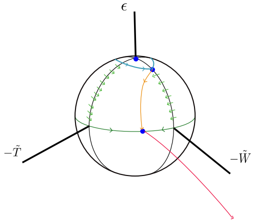


The equations in this chart:

$$\varepsilon_2' = \frac{1}{3}\varepsilon_2 W_2 (W_2 + \frac{1}{2}\varepsilon_2^2 c^2)$$
$$W_2' = 1 + \frac{2}{3} W_2^2 (W_2 - \frac{1}{2}\varepsilon_2^2 c^2)$$

Change of coordinates: $\varepsilon = r_3,\ \tilde{W} = r_3^2 W_3,\ \tilde{T} = T_3 r_3^3$





The equations in this chart (Hamiltonian):

$$T_3' = -W_3^2 + \frac{1}{2} W_3 c^2$$
$$W_3' = T_3$$

Satisfying Implicit Function Theorem

- When a heteroclinic orbit exist, h_s and h_u are identical, so $\Phi(c, \varepsilon)$ is zero if and only if a heteroclinic orbit exists.
- We have shown that the $\Phi(c, \varepsilon)$ is zero along the section when $T = 1/2$.
- Also, by regaining hyperbolicity, it is shown the Φ is smooth in parameters.

The implicit function theorem is satisfied, therefore, our heteroclinic exist in the full system!

Conclusion

- Because of the implicit function theorem, we know that a heteroclinic orbit exist.
- The limit as $\varepsilon \to 0$ is equivalent to $\rho \to \infty$ in the original scaling.
- This proves our theorem stated at the beginning, that the minimum wave speed as $\rho \to \infty$, $\tilde{c} = \sqrt[3]{\frac{3}{2}}$

References

J. J. Bramburger and C. Henderson.
The speed of traveling waves in a FKPP-burgers system.
Archive for Rational Mechanics and Analysis, 241(2):643–681, May 2021.