Floating Objects at a Two Fluid Interface

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George Mason University, MEGL

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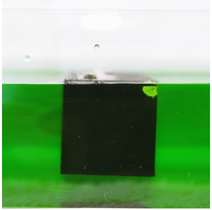
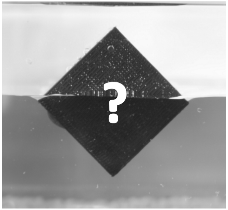
Results from previous semesters showed our ability to predict the way certain objects will float in a single fluid.

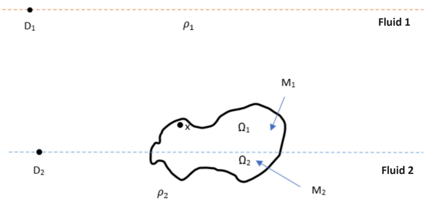
What about a two-fluid case? . . .





Motivation

- We explore the stable floating configurations of 3D printed objects at a two-fluid interface.
- All floating shapes have two-dimensional constant cross sections.
- Models are based on mathematical ideas of center of gravity, center of buoyancy, and Archimedes' Principle.
- Experimentation is used to corroborate theoretical predictions.

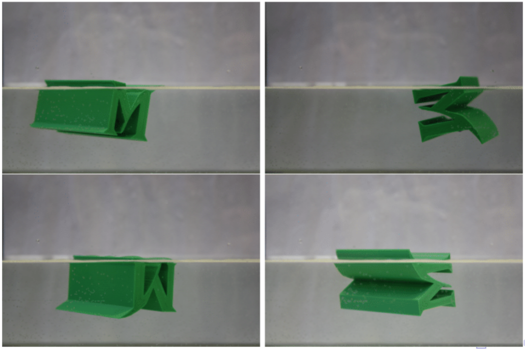


Archimedes' Principle

$$M_{obj}g = \rho_f V_{sub}g$$

Potential Energy

$$U(\theta) = M_{obj}g\,\hat{n}(\theta) \cdot (\vec{G} - \vec{B}(\theta))$$

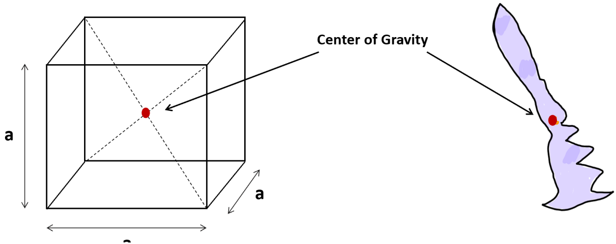


Center of Gravity

Definition

The center of gravity is a point at which the total weight of a body is concentrated. Assuming a constant cross-section L and density, the center of gravity can be written as:

$$\vec{G} = (G_x, G_y) = \frac{L}{M_{\text{obj}}} \iint_{\Omega_T} (x, y)\rho(x, y)dA$$

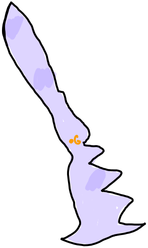


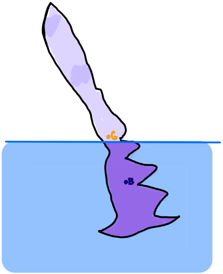
Centroid

Definition

Let $\vec{B}_i$ be the centroid for Ω_i. The centroid for region Ω_i is defined as:

$$\vec{B}_i = \frac{1}{A_i} \iint_{\Omega_i} (x, y)\,dA$$





Center of Buoyancy

Definition

Let Ω_T be the region of the submerged object. Additionally, let ρ_f be the density function for the displaced fluids. The center of buoyancy for Ω_T in the two-fluid problem can be written as:

$$\vec{B} = \frac{L}{M_1 + M_2} \iint_{\Omega_T} (x, y)\rho_f(x, y)dA$$

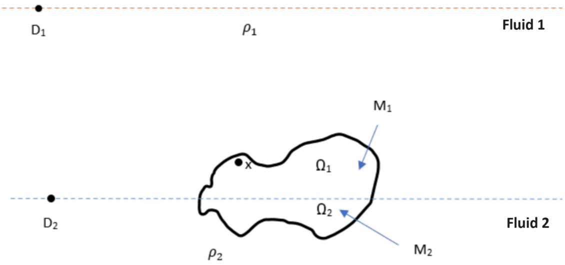
Note that if the density of the fluid in Ω_T constant the centroid and center of buoyancy in Ω_T are **equivalent**.

Relevance

Definition

Archimedes' Principle states that the buoyant force on a submerged object is equivalent to the weight of the displaced fluid. In the two-fluid scenario we can write that as:

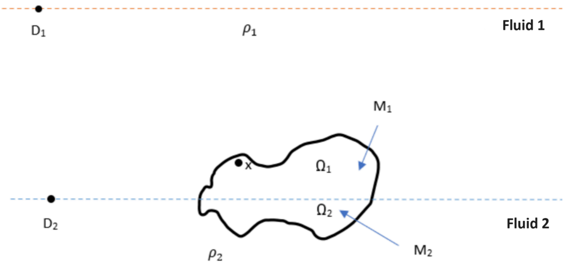
$$M_{\text{obj}}g = \rho_1 V_1 g + \rho_2 V_2 g$$



Definition

The potential energy in the Two-Fluid problem can be written as:

$$U(\theta) = M_1 g \hat{n} \cdot (\vec{G} - \vec{B_1}) + M_2 g \hat{n} \cdot (\vec{G} - \vec{B_2})$$



Derivations

Definition

From the potential function described in the previous slide the potential energy function in the two-fluid problem can be re-written as:

$$\begin{aligned} U(\theta) &= gM_{\text{obj}}(1 - \rho_1/\rho_{\text{obj}})\hat{n} \cdot (\vec{G} - \vec{B}_2) \\ &= k\hat{n} \cdot (\vec{G} - \vec{B}_2) \\ &= k\hat{n} \cdot (\vec{G} - \vec{B}(\theta)) \end{aligned}$$

Which is just a constant k times the one-fluid potential function problem. Note that if $\rho_1 = 0$ we arrive back at the one-fluid potential function.

Density Ratios

Definition

Let R_1 be the ratio of the area of the object submerged in water divided by the area of the total object in the one-fluid case and let R_2 be the ratio defined in the same way but for the two-fluid case.

$$R_1 = \frac{A_2}{A_T} = \frac{\rho_{obj}}{\rho_2}$$

and

$$R_2 = \frac{A_2}{A_T} = \frac{\rho_{\text{obj}} - \rho_1}{\rho_2 - \rho_1}$$

Notice that $R_2 = R_1$ iff $\rho_1 = 0$. Additionally, R_1 is often referred to as the **specific gravity**.

Mapping

Definition

Let $\gamma : (0,1) \to (0,1)$ where:

$$R_2 = \gamma(R_1) = \frac{R_1 - \rho_1/\rho_2}{1 - \rho_1/\rho_2}$$

γ defines an analog from the two-fluid to one-fluid case. That is, for any two-fluid problem we can reduce it to a one-fluid problem by this mapping.

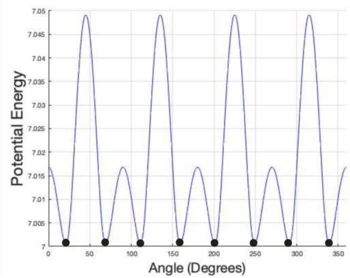
N-Fluid Extension

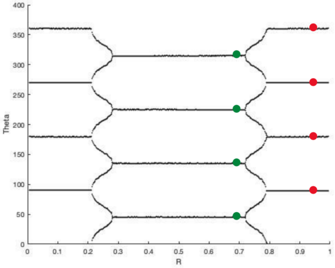
Definition

Let A_i, $\vec{B}_i$, and M_i denote the area of the object, center of buoyancy, and mass of the fluid in the ith fluid. The potential energy landscape of a floating object submerged in n-fluids can be described as:

$$U_n(\theta) = g\hat{n} \cdot \left(\sum_{i=2}^{n} M_i(\vec{G} - \vec{B}_i(\theta)) + \frac{M_1}{A_1}(1 - A_T)\vec{G} + \sum_{i=2}^{n} A_i \vec{B}_i(\theta) \right)$$

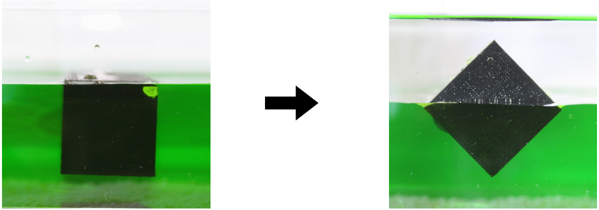
Theoretical Predictions





Given a square with a uniform cross section, the analysis above enables us to predict the angle of stable configurations for a given density ratio, R.

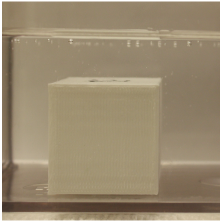
Experimentation



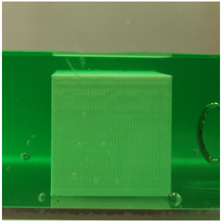
Q.E.D.

Experimentation

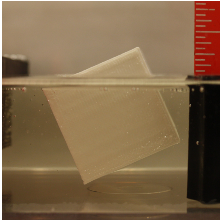
Print #	Volume & Dimension	infill %	Mass(g)	Density	R1(obj/oil)	R1(obj/water)	R1(obj/syrup)	R2 (oil-obj-water)	R2 (water-obj-syrup)	R2 (oil-obj-syrup)
52	54000	90	55.69	1.031296	1.227913	1.03627	0.751608	1.232395	0.095767	0.359647



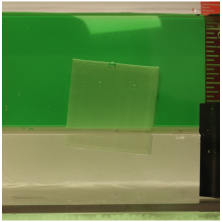
Oil



Water



Syrup

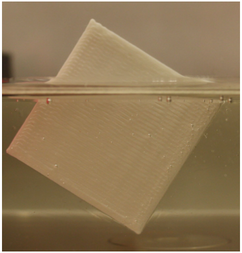


Water+Syrup

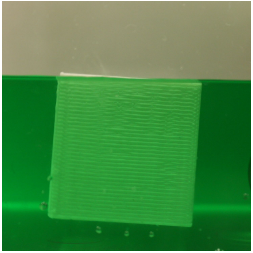
	Density
Oil	0.839877
Water	0.9952
Corn Syrup	1.372119

Experimentation

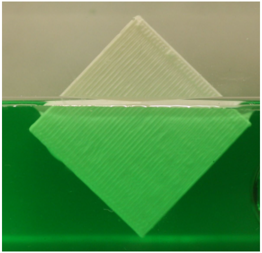
Print #	Volume & Dimension	infill %	Mass(g)	Density	R1(obj/oil)	R1(obj/water)	R1(obj/syrup)	R2 (oil-obj-water)	R2 (water-obj-syrup)	R2 (oil-obj-syrup)
50	54000	80	51.46	0.952963	1.134646	0.957559	0.694519	0.72807	-0.112059	0.212471



Oil



Water



Oil+Water

	Density
Oil	0.839877
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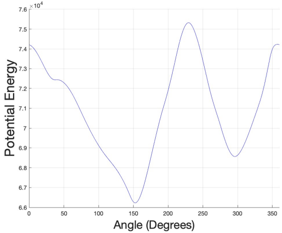
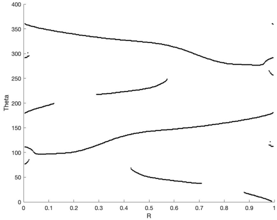
Corn Syrup	1.372119

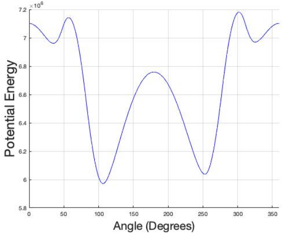
Floating GMU

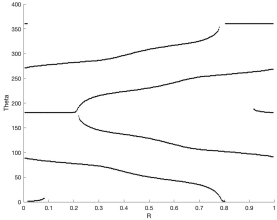
Analysis of G:





Analysis of U:





Acknowledgments

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References

1. D.M Anderson, B.G Barreto-Rosa, J.D. Calvano, L. Nsair, and E. Sander, *Mathematics of Floating 3D Printed Objects,* and Supplementary Materials: https://arxiv.org/abs/2204.08991v2

2. E.N. Gilbert, How things float, Am. Math. Mon. **98** (1991) 201–216