# Random 3d Polyforms

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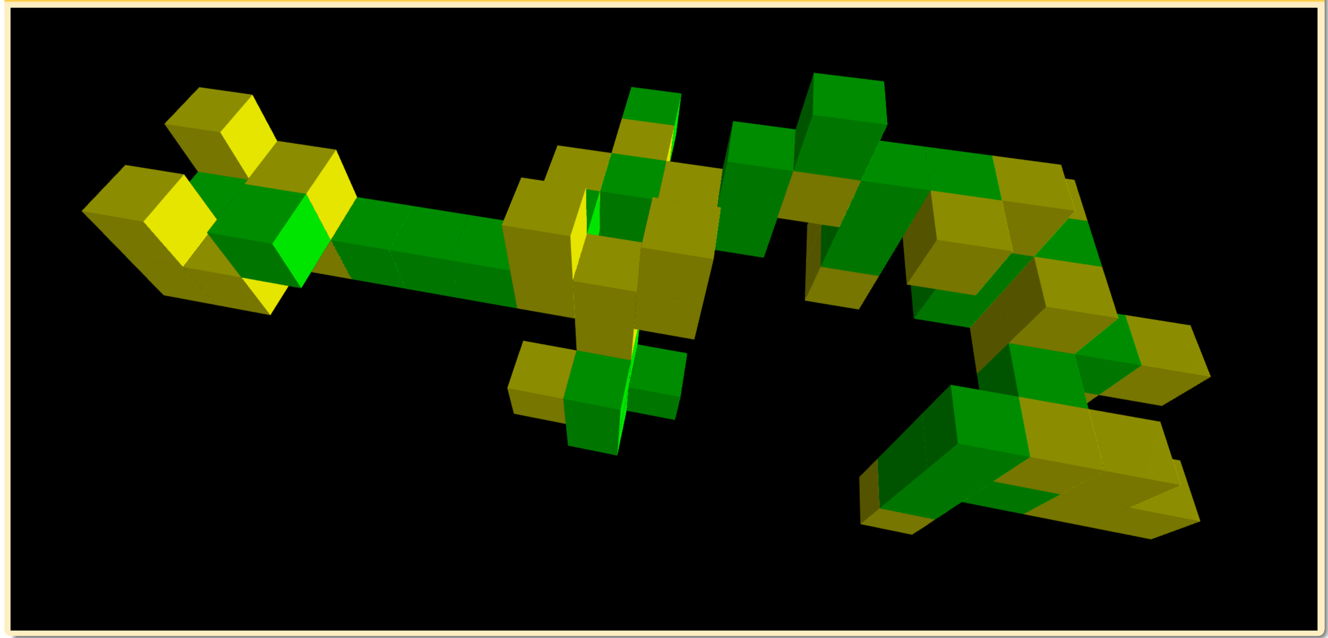
12/2/2022

## Introduction

A polyform is a figure constructed by joining together identical polytopes connected by at least one of their faces. Specifically, we are studying random polyforms composed of strongly connected cubes. Every cube is connected by a path of cubes sharing two dimensional square faces. We're interested in the homology of polyforms, such as the number of holes of such generated shapes.

## Visualization of a 3d Polyform



## Definitions

### Definition (Cubical Complex)

#### Elementary Cube

An elementary cube, $Q$ is the finite product of elementary intervals $I_i$:

$$Q = I_1 \times I_2 \times \ldots \times I_n \subset \mathbb{R}^n$$

#### Set of Cubes

The set of all elementary cubes in $\mathbb{R}^n$ by $\kappa^n$ and the set of all elementary cubes as:

$$\kappa = \bigcup_{n=0}^{\infty} \kappa^n$$

A **cubical complex** is a set of vertices, edges, squares (faces), cubes, and their n-dimensional counterparts.

### Definition (Betti Numbers)

#### Definition

We can think of the $k$th Betti number as the number of $k$-dimensional holes of a topological surface, i.e the rank of the homology group.

In the example of the torus, $b_0$ is the number of connected components (1), $b_1$ is the number of 1d holes (2), and $b_2$ is the number of 2d voids (1).

## The Work Flow

1. Generate a random point cloud with n-number of points.
2. Use the MH Algorithm to shuffle the points m-times.
3. Use the point cloud to generate the cubical toplex and construct the faces and sufaces from the top dimensional cube.
4. Use tools in Persistence Theory to compute the homology of the generated cubical complex.
5. Study how the homology changes with respect to how the polyforms are generated.

### What about other distributions?

We can move simulate a target density $\pi_p$ by modifying the previously mentioned MH-Algorithm. We use the conditional density given by the following:

$$p(A_1, A_2) = \min\{(1-p)^{t_2 - t_1}, 1\}$$

where $t_1$ and $t_2$ are the site perimeters of $A_1$ and $A_2$ respectively. At p = 1, we reject every valid polyform immediately, and thus no homology is generated. At p = 0, we accept every valid polyform, thus sampling from the uniform distribution.

## More Work

### The Metroplis Hasting Algorithm

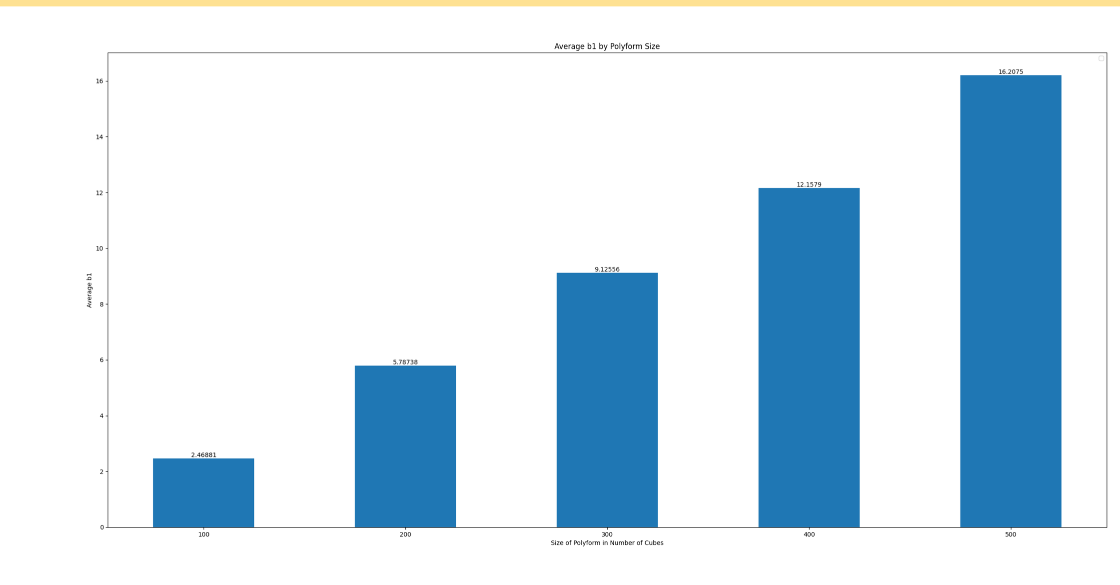
1. Set $A = L$, where $L$ is the "line" polyform with n cubes in a row.
2. Select a cube in $A$ uniformly at random. Call it $x$.
3. Remove the cube $x$ from A creating a polyform with $n-1$ cubes, $A \setminus \{x\}$.
4. Now select cube in the site perimeter of $A \setminus \{x\}$ uniformly at random, call it $y$.
5. Place a cube at $y$. The new structure created is $(A \setminus \{x\}) + \{y\}$. If $(A \setminus \{x\}) + \{y\}$ is a valid polyform $(\beta_0 \neq 1)$ then update $A = (A \setminus \{x\}) + \{y\}$ and go to **Step 1**. If $(A \setminus \{x\}) + \{y\}$ is not valid, do not update A and return to **Step 1**.

### The Metropolis Hasting Algortihm, Modified

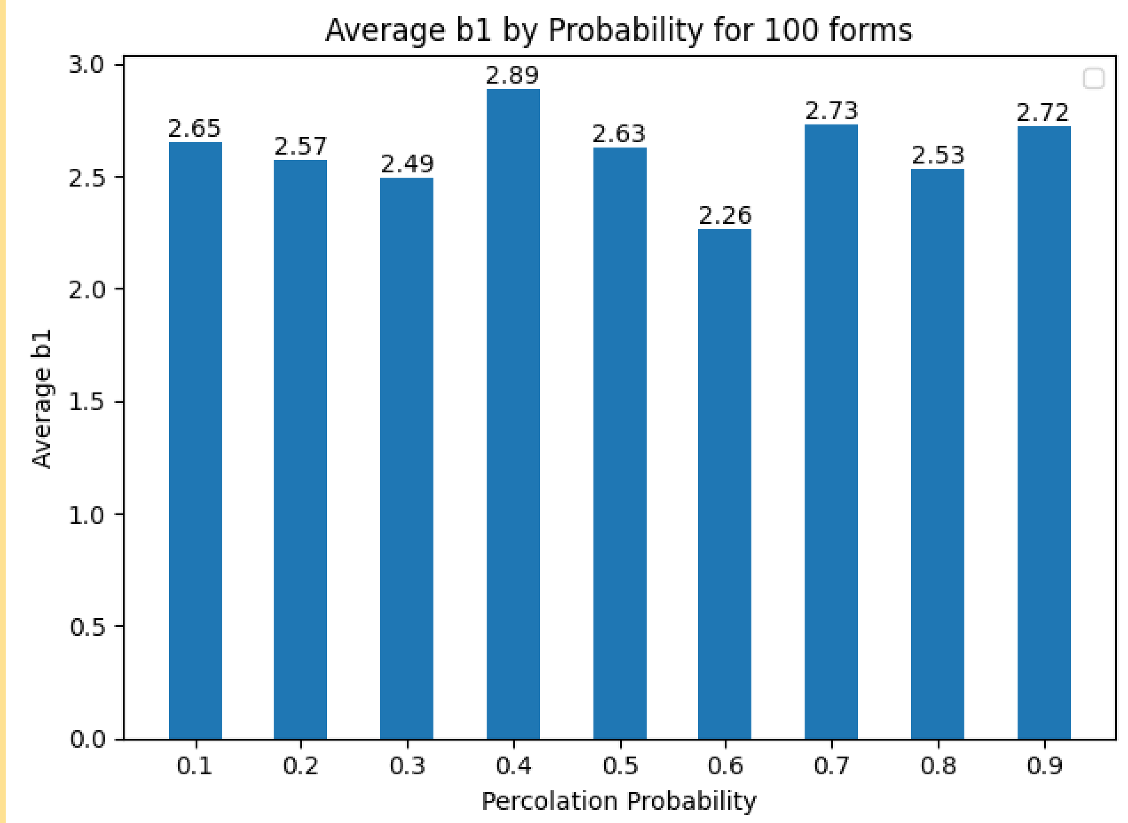
1. Set $A = L$, where $L$ is the "line" polyform with n cubes in a row.
2. Select a cube in $A$ uniformly at random. Call it $x$.
3. Remove the cube $x$ from A creating a polyform with $n-1$ cubes, $A \setminus \{x\}$.
4. Now select cube in the site perimeter of $A \setminus \{x\}$ uniformly at random, call it $y$.
5. Place a cube at $y$. Define $B = (A \setminus \{x\}) + \{y\}$. If $B$ is not a valid polyform, go to **step 1**. Else, with acceptance probability $p(A, B)$, set $A = B$ and go to **Step 1**, with rejection probability $1 - p(A, B)$, do not change $A$ and go to **Step 1**.

## Average Betti 1 Number, for Uniform Distribution and Percolation Probability

### Average Betti 1 Number by Size of Polyform



### Average Betti 1 Number for 100-forms, with percolation




## Conclusions/Future Work

We've made a lot of progress this semester, namely in computing homology, finding Big O for our shuffles, and improving the computational speed of our algorithms. Our initial data, while promising, still hasn't reached the sample sizes required to answer some key questions we have so in the future we plan on.

- Generate even larger data sets to look at limiting distributions.
- Find a pattern for the ideal number of shuffles, as mixing time is still an open problem in both the 2d and 3d case. The current guess is that mixing time in the 2d case is somewhere between $n^2$ and $n^3$ shuffles.
- Look for new and more efficient ways of shuffling polyforms, whilst still maintaining the desired distributions.

## Acknowledgments

We'd like to thank Dr. Schweinhart and Dr. Roldan for their invaluable guidance and knowledge they've provided us during this project. We'd also like to express our gratitude to Shrunal Pothagoni and Aleyah Dawkins for being our graduate mentors and MEGL for support our research. We're excited to continue next semester!

## References

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