Resolving Neural Ideals

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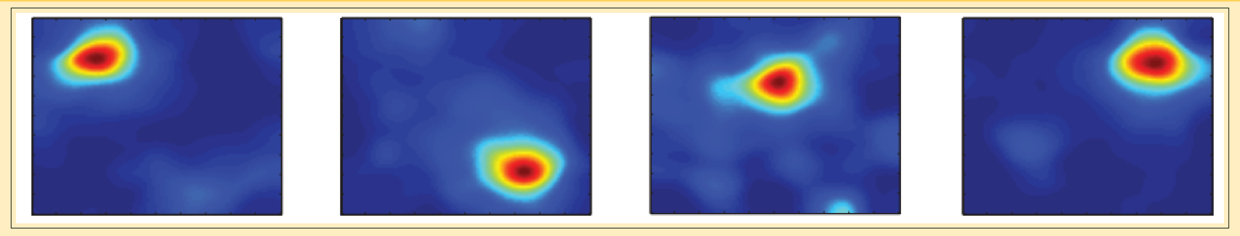
Place Fields & Stimulus Spaces [2]



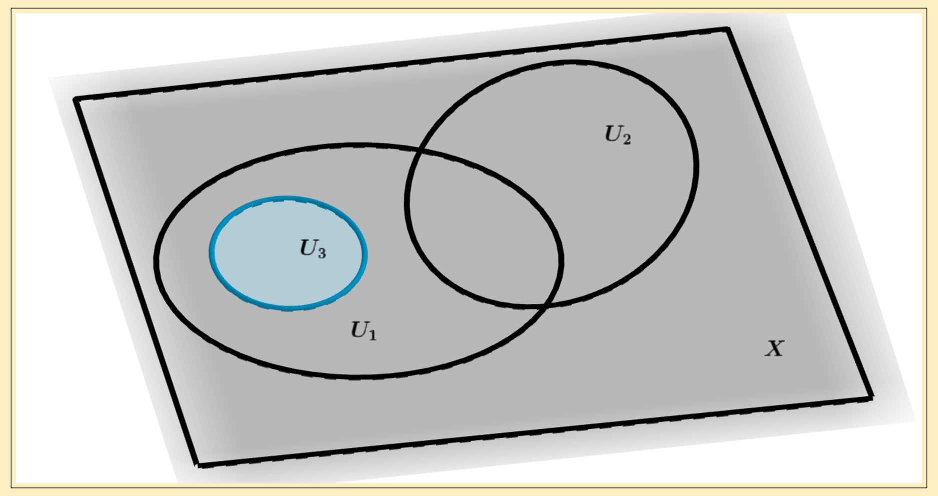
Heat map of mice place cell activity in an experiment [3]

Place Cells Class of neuron which "processes" the location of an animal in some region of space.

Stimulus Space Closed region X containing possible stimuli.

Place Field Area corresponding to a place cell which fires whenever an animal is in said area.

Receptive Field Structure [2]



RF of three neurons with an activation corresponding to a codeword $c = (1, 0, 1)$

Receptive Field Set $U_i \subset X$ corresponding to some place field of the i^{th} neuron in a population of n.

RF code Set of all binary codewords corresponding to stimuli in X. An entry in codeword $c \in \mathcal{C} \subseteq \{0,1\}^n$ takes a value of 1 when stimulated.

Neural Ideals & Pseudomonomial Relations [2]

Pseudomonomials are of the form

$$\prod_{i\in\sigma} x_i \prod_{j\in\tau}(1-x_j)$$

in the ring $R = \mathbb{F}_2[x_1, \ldots, x_n]$ where $\sigma \cap \tau = \emptyset$ and $\sigma, \tau \subset \{1, \ldots, n\}$ correspond to indices of codewords in $\mathcal{C}$.

Pseudomonomial Relations Relations on a pseudomonomial ideal corresponding to set containment types in the RF structure.

Type	Relation	Containment
T1	$\prod_{i\in\sigma} x_i$	$\bigcap_{i\in\sigma} U_i = \emptyset$
T2	$\prod_{i\in\sigma} x_i \prod_{j\in\tau}(1-x_j)$	$\bigcap_{i\in\sigma} U_i \subseteq \bigcup_{j\in\tau} U_j$
T3	$\prod_{j\in\tau}(1-x_j)$	$X \subseteq \bigcup_{j\in\tau} U_j$

Canonical Forms & Resolutions [2]

The **canonical form** of a neural ideal $J_\mathcal{C}$, denoted $\mathrm{CF}(J_\mathcal{C})$ consists of a set of irredundant pseudomonomial relations; this allows us to "read off" the minimal combinatorial structure of the neural code $\mathcal{C}$.

From the relations in $\mathrm{CF}(J_\mathcal{C})$, we can realize a **free resolution** given by

$$\mathcal{F}: 0 \longrightarrow R^{\beta_d} \xrightarrow{\partial_d} R^{\beta_{d-1}} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} \mathbb{F}_2^{\beta_1} \xrightarrow{\partial_1} R \longrightarrow R/J_\mathcal{C} \longrightarrow 0$$

where $1, \beta_1, \ldots, \beta_d$ is a **Betti sequence** or sequence of **Betti numbers** which hopefully allow us to classify differents types of neural ideal canonical forms and, therefore, different types of RF structures.

Canonical Form classes of Polarized Neural Ideals [1,4]

Along with some miscellaneous (i.e. randomly generated) neural ideals, we primarily inspected the resolutions of two types (families) of canonical forms from [1, Theorems 7.1 & 7.4] in some smaller cases. These results depend on the polarization of a neural ideal, $\mathcal{P}(J_\mathcal{C})$ and its corresponding canonical form.

The **polarization** of a pseudomonomial in $J_\mathcal{C}$ from R to $R[y_1, \ldots, y_n]$ is given by

$$\mathcal{P}\left(\prod_{i\in\sigma} x_i \prod_{j\in\tau}(1-x_j)\right) = \prod_{i\in\sigma} x_i \prod_{j\in\tau} y_j$$

and $\mathcal{P}(J_\mathcal{C})$ is an element-wise polarization of $J_\mathcal{C}$'s generators.

Example

(Theorem 7.1[1]; length 2 example) The canonical form of a polarized neural ideal in $K = R[y_1, \ldots, y_7]$ given by

$$\langle x_1g_1, x_2y_1g_2, y_2, x_2[g_1g_2], [g_1g_2g_3], y_1[g_2g_3]\rangle$$

where $g_1 = x_4x_5x_6$, $g_2 = x_4x_7$, $g_3 = x_4x_6x_7$ and $[g_p, g_q]$ denotes the LCM of g_p and g_q with the g_k's indicating specification on a family of canonical forms of this type. This yields the resolution

$$0 \longrightarrow K \longrightarrow K^5 \longrightarrow K^5 \longrightarrow K \longrightarrow 0$$

which has the Betti sequence 1,5,5,1.

(Theorem 7.1[1]; length 3 example) The canonical form of a polarized neural ideal in $K = R[y_1, \ldots, y_8]$ given by

$$\langle x_1g_1, x_2[g_1g_2], x_3[g_1g_2g_3], [g_1g_2g_3g_4], x_2y_1g_2, x_3y_1[g_2g_3], y_1[g_2g_3g_4], x_3y_2g_3, y_2[g_3g_4], y_3g_4\rangle$$

where $g_1 = g_3 = x_6x_7x_8$, $g_2 = x_5$, $g_4 = x_5x_7x_8$. We receive the resolution

$$0 \longrightarrow K^2 \longrightarrow K^6 \longrightarrow K^5 \longrightarrow K \longrightarrow 0$$

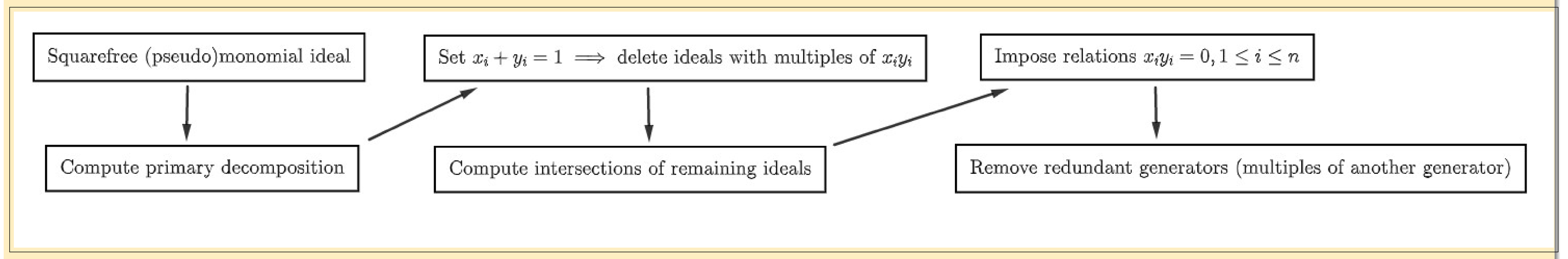
with the Betti sequence 1,5,6,2.

Similar with examples from Theorem 7.4, one of our primary goals was to look for consistent characteristics of Betti sequences depending on how choices of $g_1, \ldots, g_s$ related to one another (common divisors, equality, etc.).

Computation [1]

Computations on a large amount of ideals of the aforementioned classes were done in Macaulay2. We computed canonical forms using an implementation of Algorithm 3.2 from [1], which is an improvement upon the original in [2], but in terms of polarized neural ideals.

Many scripts were also written with the intent to generate large amounts of random configurations of certain canonical form classes, as well as specific classes such as the previously mentioned Theorems 7.1 and 7.4.



Visual summary of Algorithm 3.2 [1]

Results, Conjectures & Future Directions

- We found certain examples of Theorems 7.1 and 7.4 from [1] where we receive consistent Betti sequences which are symmetric, but do not take the form $\binom{\ell}{0}, \binom{\ell}{1}, \ldots, \binom{\ell}{\ell}$, where ℓ is the number of relations in $\mathrm{CF}(J_\mathcal{C})$ (i.e. not Taylor resolutions). Similarly, we wonder of Betti sequences which are not symmetric and not binomial coefficients, but appear to yield the same Betti sequence for different choices of $g_1, \ldots, g_s$. We conjecture some predictable characteristics that hold these Betti sequences as algebraic invariants, thus allowing us to characterize the corresponding RF structures.
- We wish to do the same for other classes as defined in Section 7 of [1].
- We aim to compile the various scripts (and more) into a package for the computation and analysis of neural ideals with Macaulay2.

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