Hyperbolicity and Heteroclinic Orbits

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Introduction

We are interested in heteroclinic orbits for the following system when $\rho \to \infty$

$$T' = -\tilde{c}T + UT + \frac{1}{2\rho}U(2\tilde{c} - U) \quad (1)$$

$$U' = \left(\frac{\rho}{U - \tilde{c}}\right) T(1 - T) \quad (2)$$

Change variables and desingularize using $\varepsilon^3 = \frac{1}{\rho}$, $\tilde{c} = \frac{c}{\varepsilon}$, and $W = U\varepsilon$ to obtain.

$$\dot{T} = -T(W - c)^2 + \frac{1}{2}W\varepsilon^2(2c - W)(c - W) \quad (3)$$

$$\dot{W} = -T(1 - T). \quad (4)$$

Heteroclinic orbits with wave speeds $\tilde{c} \geq c^*(\rho)$ correspond to traveling waves in the solution.(Bramburger & Henderson). A minimum wave speed for the system under the ρ scaling is bound in the below theorem. The theorem presented here is that the minimum wave speed is the lower bound.

Theorem (Bramburger & Henderson 2021)

$$\sqrt[3]{\frac{3}{2}} \leq \liminf_{\rho\to\infty} \frac{c_*(\rho)}{\rho^{1/3}} \leq \limsup_{\rho\to\infty} \frac{c_*(\rho)}{\rho^{1/3}} \leq \sqrt{3} \quad (5)$$

Theorem (HKMTW)

$$\lim_{\rho\to\infty} \frac{c_*(\rho)}{\rho^{1/3}} = \sqrt[3]{\frac{3}{2}} \quad (6)$$

Implicit Function Theorem to Prove Existence of Heteroclinic Orbits

A new function, $\tilde{c}^*(\varepsilon)$, can be found if three things are proven: First, Let

$$\Phi(c, \varepsilon) = h_u(c, \varepsilon) - h_s(c, \varepsilon)$$

Where h_s is a solution curve which passes through $(0, 0)$ and h_u is a solution curve which passes through $(1, c)$.

- 1. Show some function, $\Phi(c, \varepsilon)$, can be constructed where $\Phi(c^*, 0) = 0$ along a section Σ where $\Sigma = \{(T, W) \mid T = \frac{1}{2}\}$
- 2. Show the parameters c and ε are smooth along those sections
- 3. Show that $\frac{\partial\Phi}{\partial c}(c^*, 0) \neq 0$ along the section Σ

Heteroclinic Orbit in Reduced System ($\epsilon = 0$)

Solution curve from an unstable fixed point to a stable fixed point

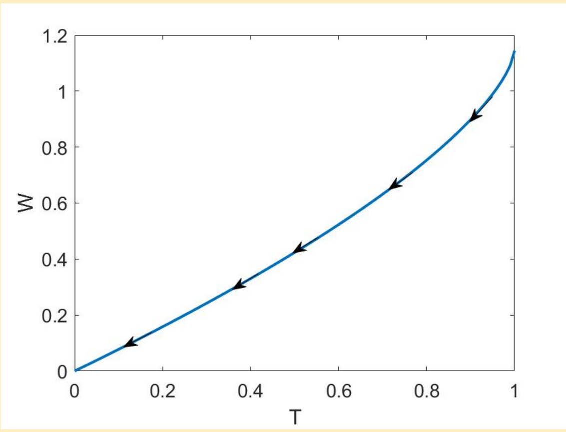


Figure: Solution of ODE

$$\frac{dW}{dT} = \frac{1 - T}{(W - \tilde{c})^2} \quad (7)$$

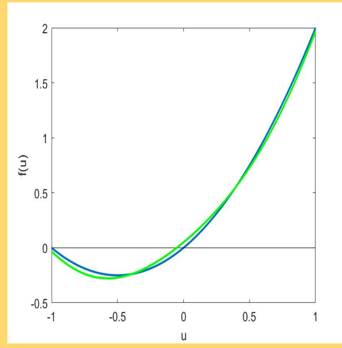
We can solve the separable ODE to find $\tilde{c} = \sqrt[3]{\frac{2}{3}}$ when $\varepsilon = 0$. So if a heteroclinic orbit exists, then $\tilde{c} = \sqrt[3]{\frac{3}{2}}$.

Hyperbolicity

Hyperbolic fixed points used to apply knowledge of linear systems to nonlinear systems.

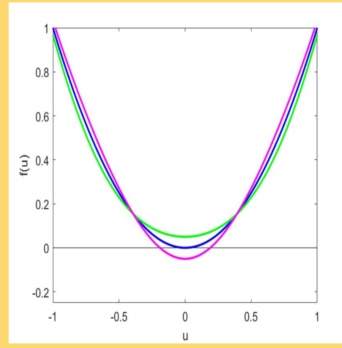
1-D Examples

Hyperbolic



With hyperbolic fixed points, reduced linear dynamics approximate the non-linear dynamics

Non-Hyperbolic



With non-hyperbolic fixed points, reduced linear may not approximate reduced linear dynamics

Blow-up

The fixed point $(1, c)$ is not hyperbolic. We will use geometric blow-up (desingularization) to recover hyperbolicity.

To do the blow up, a simple change of basis is done to translate the origin of our equations. $\tilde{T} = T - 1$ and $\tilde{W} = W - \tilde{c}$ The equations change to

$$\dot{\tilde{T}} = -\tilde{W}^2(\tilde{T} + 1) - \frac{1}{2}\tilde{W}\varepsilon^2(c^2 - \tilde{W}^2)$$

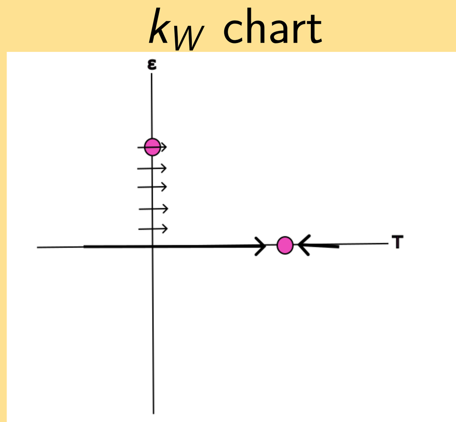
$$\dot{\tilde{W}} = \tilde{T}(\tilde{T} + 1)$$

The blow up is a quasi-homogeneous blow up that will blow up our fixed point into an ellipsoid. (Alverez 2011)

Charts

Three coordinate charts used to analyze non-hyperbolic fixed point.

k_W chart



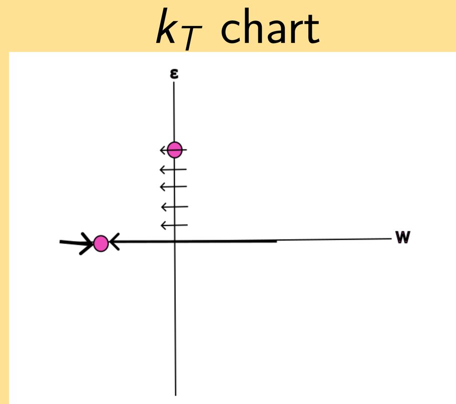
Change of coordinates:

$$\tilde{T} = r_1^3 T_1$$
$$\tilde{W} = -r_1^2$$
$$\varepsilon = r_1\varepsilon_1$$

Resulting equations:

$$T_1' = \frac{3}{2}T_1 + 1 - \frac{\varepsilon_1 c^2}{2}$$
$$\varepsilon_1' = \frac{\varepsilon_2 T_1}{2}$$

k_T chart



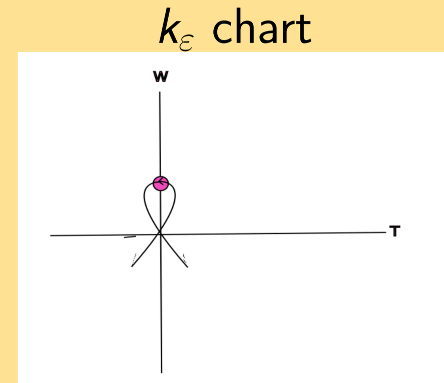
Change of coordinates:

$$\tilde{T} = r_2^3$$
$$\tilde{W} = r_2^2 W_2$$
$$\varepsilon = r_2\varepsilon_2$$

Resulting equations:

$$\varepsilon_2' = \frac{1}{3}\varepsilon_2 W_2(W_2 + \frac{1}{2}\varepsilon_2^2 c^2)$$
$$W_2' = 1 + \frac{2}{3}W_2^2(W_2 - \frac{1}{2}\varepsilon_2^2 c^2)$$

k_ε chart



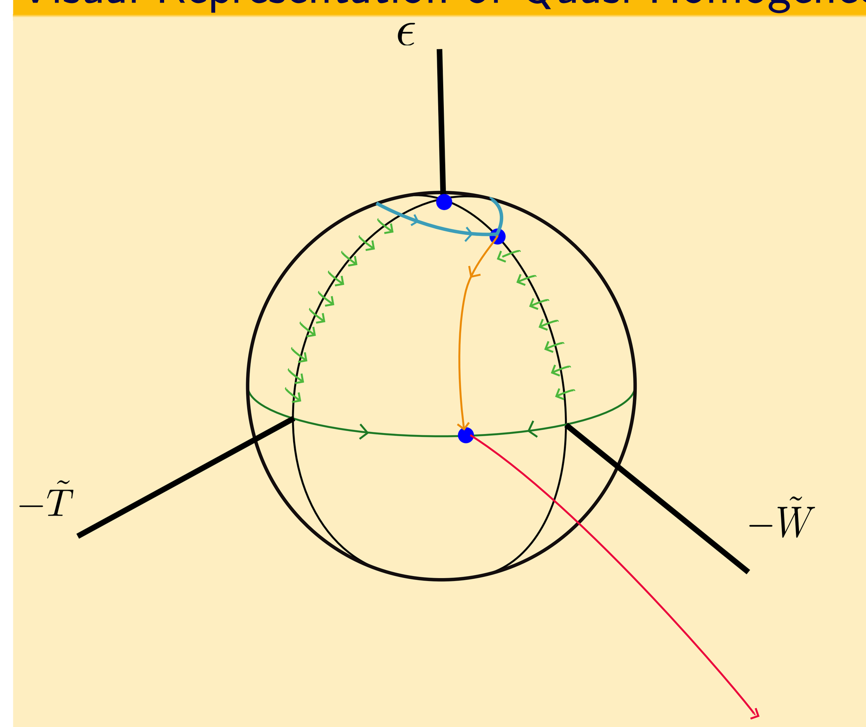
Change of coordinates:

$$\tilde{T} = r_3^3 T_3$$
$$\tilde{W} = r_3^2 W_3$$
$$\varepsilon = r_3$$

Resulting equations (Hamiltonian):

$$T_3' = -W_3^2 + \frac{1}{2}W_3 c^2$$
$$W_3' = T_3$$

Visual Representation of Quasi-Homogeneous Blow Up



This blow up shows the three fixed points that were truncated when the small parameter, ε, is set to zero. By projecting onto the three charts, the top two fixed points are in a Hamiltonian system are found in the k_ε chart and the remaining fixed point is found in the k_T chart. The green arrows, along with the diagrams of the charts, represent a trapping region for the solution that connects the two fixed points. The trapping region ensures that no solution can leave the region, thus, establishing the existence of a solution that connects the two.

Satisfying Implicit Function Theorem

When a heteroclinic orbit exists, h_s and h_u are identical, so $\Phi(c, \varepsilon)$ is zero if and only if a heteroclinic orbit exists. We have shown that the $\Phi(c, \varepsilon)$ is zero along the section when $T = 1/2$. By regaining hyperbolicity, it is shown that Φ is smooth in parameters. All conditions for the Implicit Function Theorem are met.

Conclusion

From the implicit function theorem, we know that a heteroclinic orbit exists and so we can express W as a function of T and inspect $\frac{dW}{dT}$. The limit of $\varepsilon \to 0$ is equivalent to $\rho \to \infty$ in the original system. Thus, there exist traveling wave solutions for wave speeds, $\tilde{c} \geq \sqrt[3]{\frac{3}{2}}$

References

Bramburger, J.J., Henderson, C. The Speed of Traveling Waves in a FKPP-Burgers System. Arch Rational Mech Anal 241, 643–681 (2021). https://doi.org/10.1007/s00205-021-01660-5

Alvarez, Ferragut, Jarque. A Survey on the Blow Up Technique. International Journal of Birfurcation and Chaos, Vol. 21, No. 11, 3103-3118 (2011)/. https://doi.org/10.1142/S0218127411030416