Stability of Floating Objects at a Two-Fluid Interface

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Introduction

We explore the stability of floating objects at a two-fluid interface through mathematical modeling and experimentation. Our models are based on the ideas of center of gravity, center of buoyancy, and Archimedes' Principle. We investigate floating shapes with two-dimensional constant cross sections and calculate a potential energy landscape that helps identify stable and unstable floating orientations. We compare our analyses and computations to experiments designed and created through 3D printing.

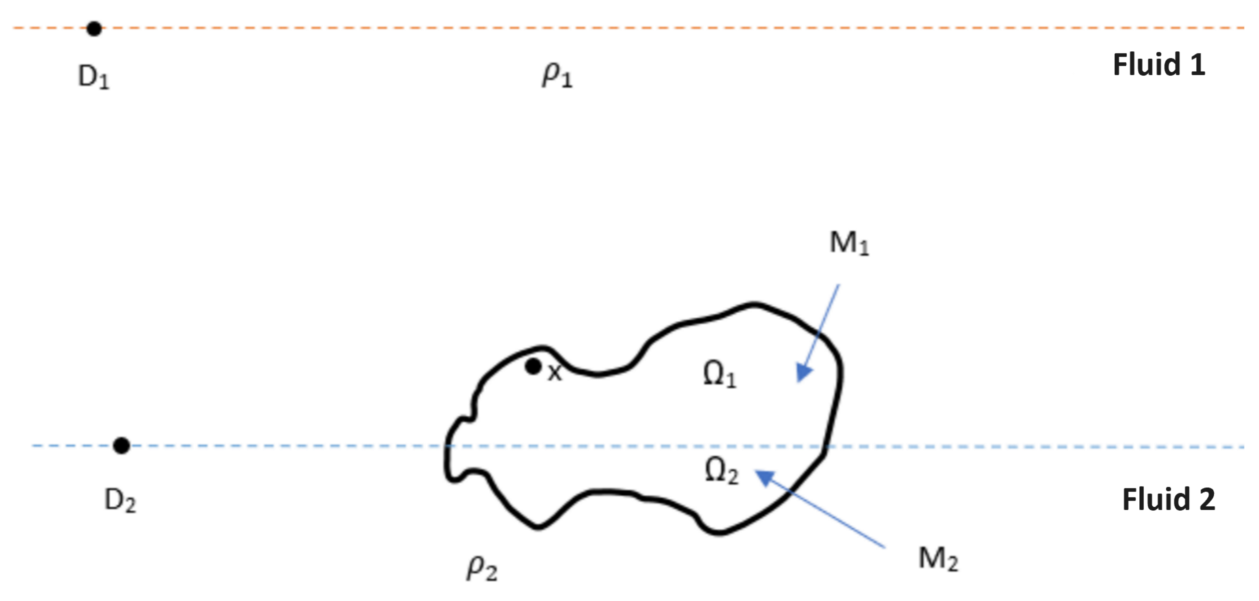

Figure 1. Object at a two-fluid interface.

Center of Gravity

Let L be the length of the cross-section, M_{obj} object mass, and $\rho(x, y)$ be the density function of the submerged object.

$$\vec{G} = (G_x, G_y) = \frac{L}{M_{\text{obj}}} \iint_{\Omega_T} (x, y)\rho(x, y)dA$$

Center of Buoyancy

Let Ω_T be the region of the submerged object, ρ_f be the density function for the displaced fluids. The center of buoyancy for Ω_T in the two-fluid case can be written as:

$$\vec{B} = \frac{L}{M_1+M_2} \iint_{\Omega_T} (x, y)\rho_f(x, y)dA$$

Centroid

Let $\vec{B}_i$ be defined as the centroid for region Ω_i:

$$\vec{B}_i = \frac{1}{A_i} \iint_{\Omega_i} (x, y)dA$$

Extended Definition of Center of Gravity

$$\vec{G} = \frac{1}{A_T}(A_1\vec{B}_1 + A_2\vec{B}_2)$$

Archimedes Principle

The buoyant force acting on the object is equivalent to the weight of the fluid in region Ω_1 plus the weight of the fluid in region Ω_2. This can be written as:

$$\rho_1 A_1 Lg + \rho_2 A_2 Lg = \rho_{\text{obj}} A_T Lg$$

Potential Energy

One-Fluid

The potential energy landscape is the sum of the work against gravity to displace an object plus work done against gravity on the displaced fluids.

$$U = U_G + U_B$$

Allows us to represent potential energy of the one-fluid problem as:

$$U(\theta) = M_{\text{obj}} g \hat{n}(\theta) \cdot (\vec{G} - \vec{B}(\theta))$$

Two-Fluid

Let $\vec{B}_1$, M_1 be the center of buoyancy and mass of the displaced fluids in region Ω_1. Define $\vec{B}_2$ and M_2 in a similar manner. The potential energy landscape for the two-fluid problem is:

$$U(\theta) = M_1 g \hat{n} \cdot (\vec{G} - \vec{B}_1) + M_2 g \hat{n} \cdot (\vec{G} - \vec{B}_2)$$

Potential Energy Two-Fluid (Continued)

Simplifying the two-fluid case reflects a scalar multiple of the potential energy function in the one-fluid case.

$$\begin{aligned} U(\theta) &= gM_{\text{obj}}(1 - \rho_1/\rho_{\text{obj}})\hat{n} \cdot (\vec{G} - \vec{B}_2(\theta)) \\ &= k\hat{n} \cdot (\vec{G} - \vec{B}_2(\theta)) \end{aligned}$$

Let $\gamma : (0, 1) \to (0, 1)$ be a bijective function. Stable orientations of the two-fluid problem are equivalent to the stable orientations of the one-fluid problem by the following mapping.

$$\gamma(R_1) = \frac{R_1 - \rho_1/\rho_2}{1 - \rho_1/\rho_2}$$

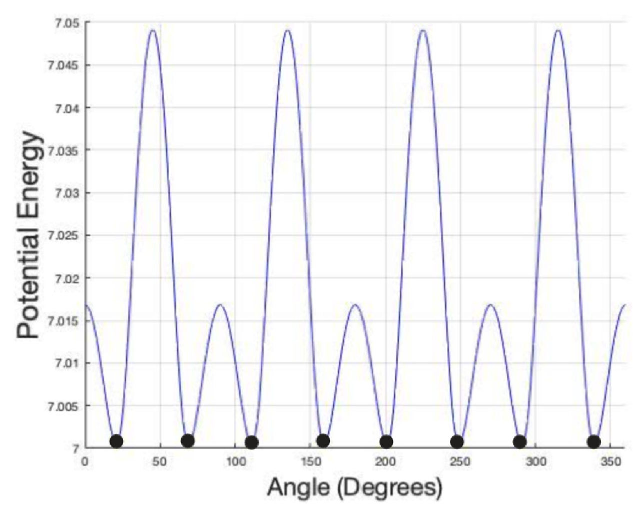

Figure 2. Potential energy landscape for the square. The minima of the function represent predicted stable orientations at a corresponding angle.

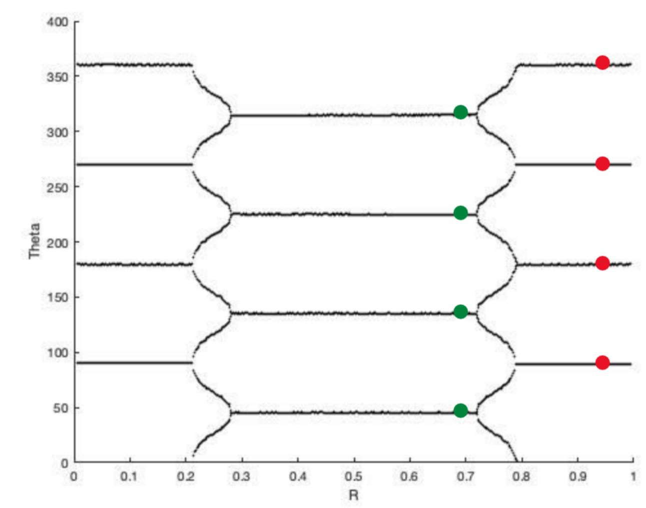

Figure 3. Stable equilibrium floating orientations for the square. The green and red dots indicate density ratios R_1 and R_2, respectively.

Relative Density

One-Fluid

The relative density of the one-fluid is the ratio of the density of the submerged object to the density of the fluid.

$$R_1 = \frac{\rho_{\text{obj}}}{\rho_2}$$

Two-Fluid

The relative density for the two-fluid problem can be represented as:

$$R_2 = \frac{A_2}{A_T} = \frac{\rho_{\text{obj}} - \rho_1}{\rho_2 - \rho_1}$$

Conclusions/Future Work

Experimentation has corroborated our theoretical results in predicting stable floating configurations of an object with a uniform cross section at a two-fluid interface. The mathematical components of the two-fluid case enables us to transform it into a model that simply scales the one-fluid case. We were able to utilize work from previous semesters to analyze this new interface. The yet explored implications of surface tension on the potential energy analysis leave questions for future semesters' work.

Figure 4. GMU Letters floating at different fluid interfaces.

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