Analyzing Monotonicity in the Linearized S.I.R. Model

Raina Joy Saha, Caleb Schear, Marcelo Montañez-Collado
Dr. Holzer, John B. Kent

Mason Experimental Geometry Lab

George Mason University

May 14, 2022

Abstract

We model arrival times of an infection in a network and look at monotonicity of these times as a function of a diffusion parameter (the propensity for population to travel between nodes). Using the limiting models of infinite lattice graphs, we identify some special cases and conditions for nonmonotonicity.

Introduction

Networks allow modeling of uneven population densities. To determine the probability of a random walk of size n between nodes, we can look at the powers of the adjacency matrix of the network. In this project, we study the relationships between walk length and probability and determine what causes decreasing and non-monotone behavior in probability and spread speed as we increase the travel rate of people.

Linearization the extended SI model

The SIR model is governed by this series of differential equations, where P is a matrix representing the connections between nodes as probabilities of "traversal", uniform on each node:

$$\frac{dS_n}{dt} = -\alpha I_n S_n + \gamma \sum_m P_{nm}(S_m - S_n) \tag{1}$$

$$\frac{dI_n}{dt} = \alpha I_n S_n - \beta I_n + \gamma \sum_m P_{nm}(I_m - I_n) \tag{2}$$

Which can be linearized by $(S, I) = (1 - s, i)$,

$$\frac{di_n}{dt} = (\alpha - \beta - \gamma) i_n + \gamma \sum_m P_{nm} i_n \tag{3}$$

Matrix solution

$$\frac{d\mathbf{i}}{dt} = ((\alpha - \beta - \gamma)\mathbb{I} + \gamma P)\mathbf{i} \tag{4}$$

Then apply the substitution $\{\tau = (\alpha - \beta)t, \tilde{\gamma} = \frac{\gamma}{\alpha - \beta}\}$ and rename variables appropriately to simplify:

$$\frac{d\mathbf{i}}{dt} = ((1 - \gamma)\mathbb{I} + \gamma P)\mathbf{i} \tag{5}$$

We look at when a specific node breaches an arbitrary threshold infection of 1 and expand the matrix exponential.

$$i_n(t) = F(\gamma, t) = e^{(1-\gamma)t} \sum_{k \geq 0} \frac{P_k}{k!} \gamma^k t^k = 1 \tag{6}$$

The total derivative of the equation

The total derivative of the equation can be determined to be

$$\frac{dt}{d\gamma} = -\frac{F_\gamma}{F_t} \tag{7}$$

by solving for the partial derivative of equation (6) with respect to γ and t we will get the equations.

$$F_\gamma = e^{(1-\gamma)t} \sum_{k \geq 0} \frac{t^{k+1}\gamma^k}{k!}(P_{k+1} - P_k)$$

$$F_t = e^{(1-\gamma)t} \sum_{k \geq 0} \frac{t^k \gamma^k}{k!}(P_k + \gamma(P_{k+1} - P_k))$$

GUI Results and Analysis

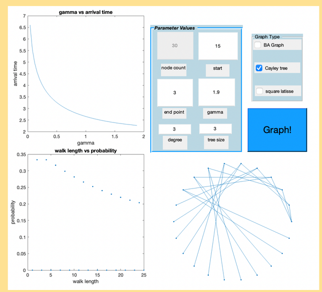

Figure: GUI created on Matlab. creates a probability vs arrival time and a gamma vs arrival time graph for the selected graph type.

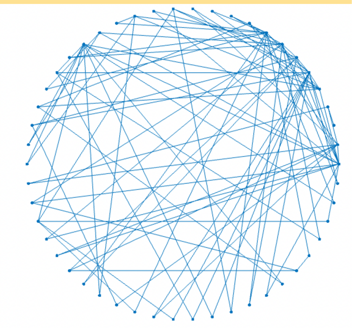

Figure: Barabási–Albert graph

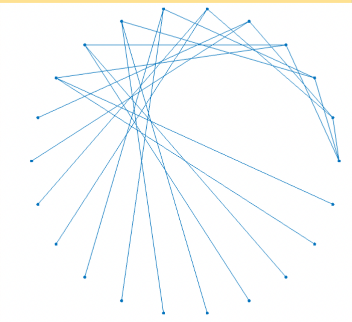

Figure: Caylee Tree graph

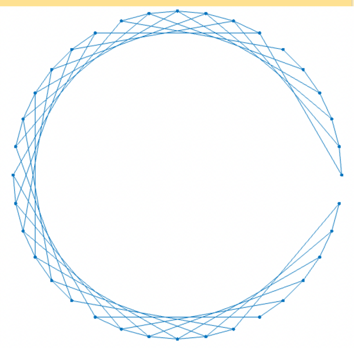

Figure: Square Lattice graph

Graphs

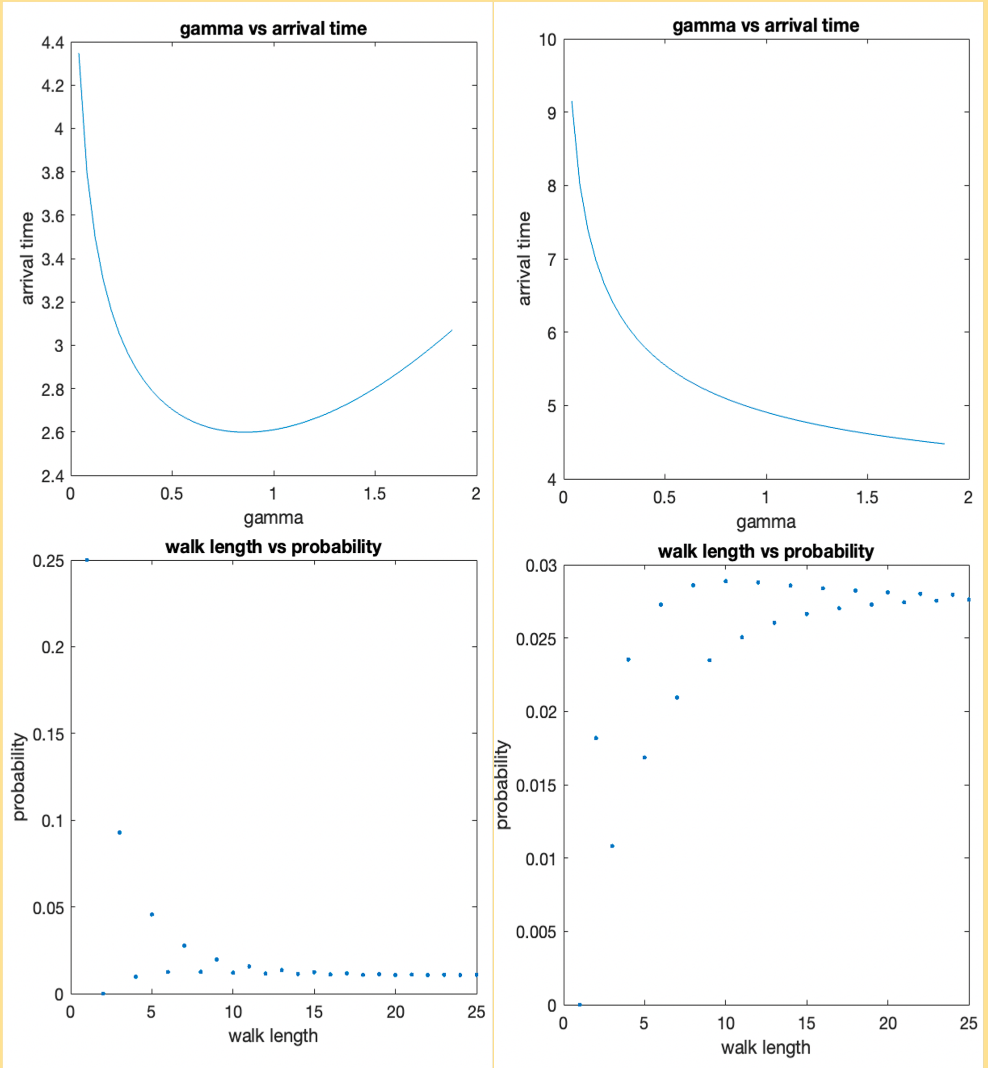

Figure: probability vs arrival time and γ vs arrival time graphs for the non-monotone and monotone decreasing BA graphs

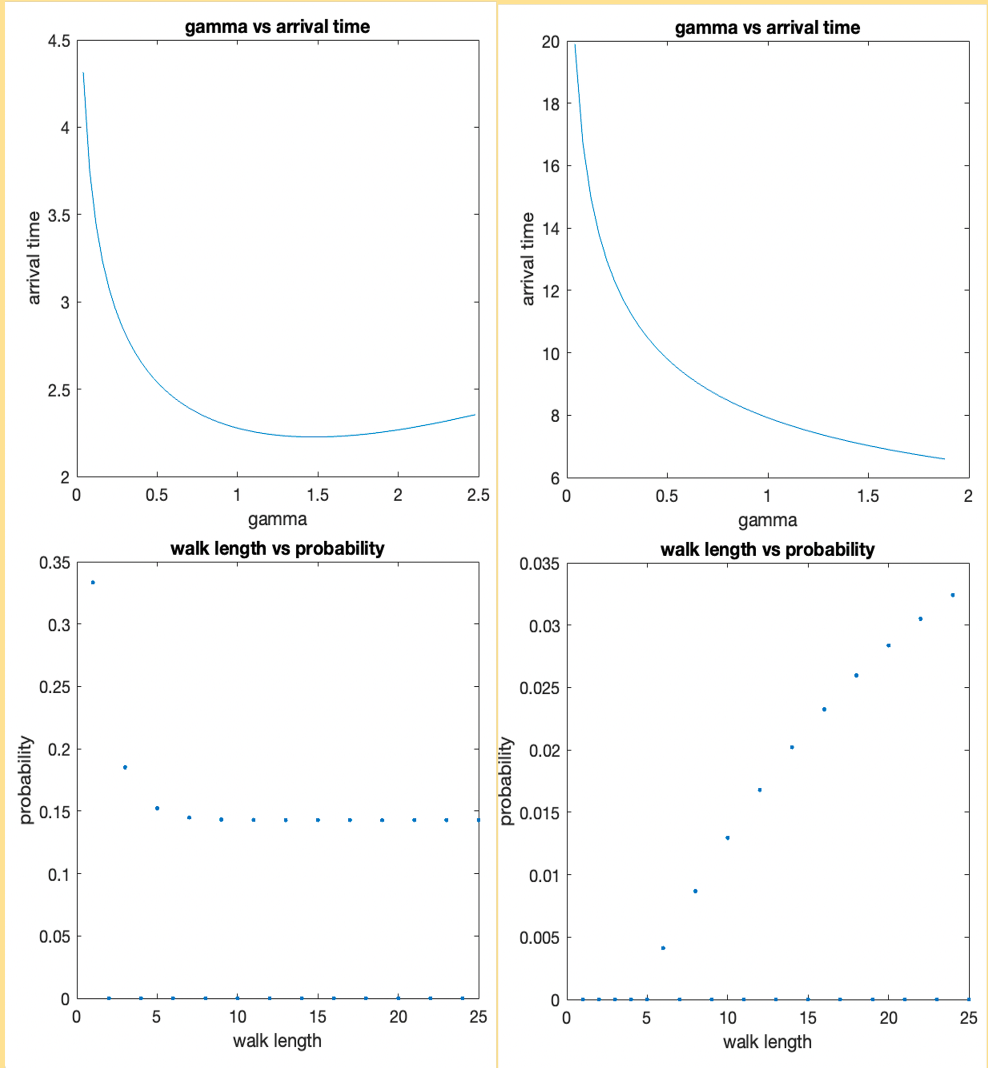

Figure: probability vs arrival time and γ vs arrival time graphs for the non-monotone and monotone decreasing Caylee Tree graphs

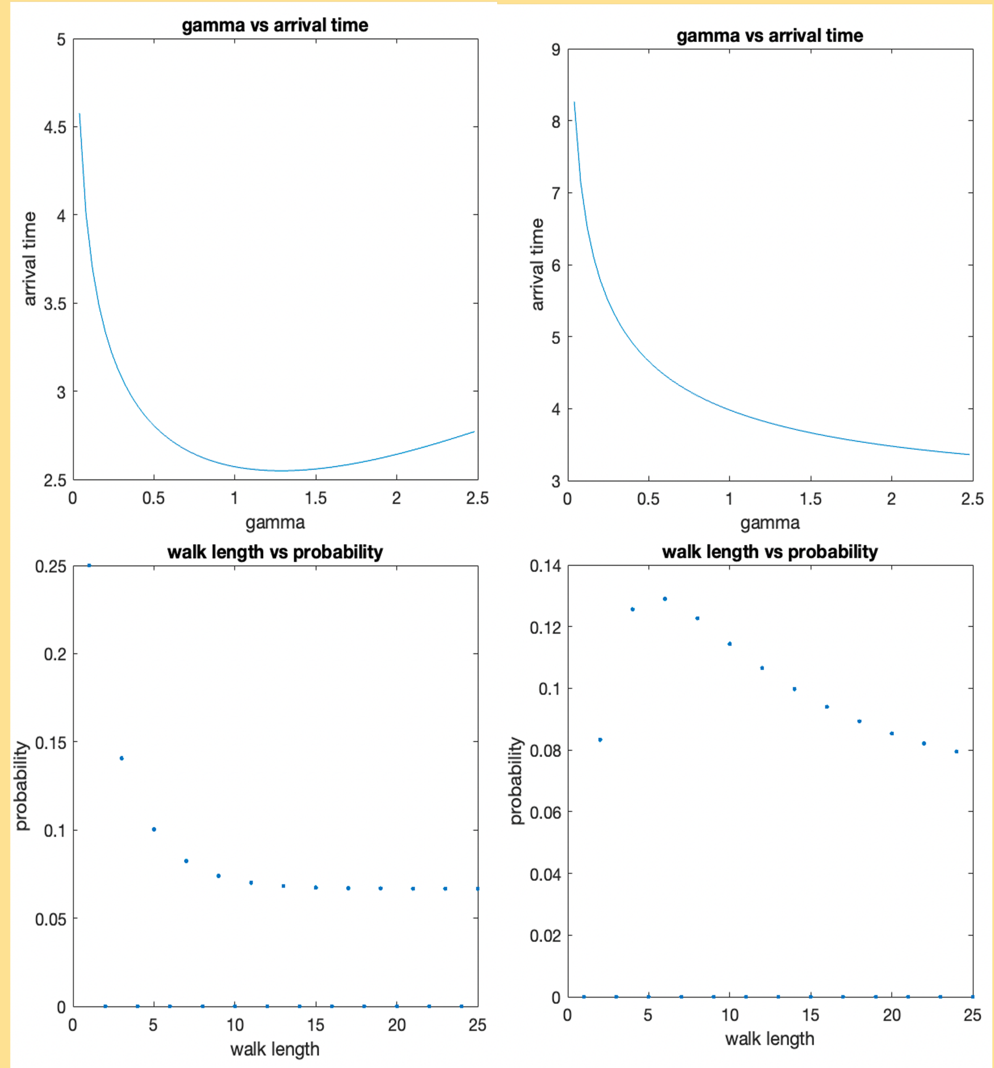

Figure: probability vs arrival time and γ vs arrival time graphs for the non-monotone and monotone decreasing Square lattice graphs

monotone decreasing and non-monotone

What makes a graph monotone decreasing vs non-monotone can be determined by looking at the equation (7). The signs of $-P_k + P_{k+1}$ and $P_k + \gamma(P_{k+1} - P_k)$ control the sign of $\frac{dt}{d\gamma}$ and thus whether it is monotone decreasing or non-monotone

- $P_{k+1} > P_k$
  F_γ and F_t are also always positive. Thus monotone decreasing.
- $P_{k+1} < P_k$
  F_γ will always be negative. However, the sign of equation F_t depends on γ.
  - $\gamma < \frac{-P_k}{(P_{k+1} - P_k)}$
    F_t is positive and the the graph is non-monotone
  - $\gamma > \frac{-P_k}{(P_{k+1} - P_k)}$
    F_t is negative and the the graph is monotone
  - $\gamma = \frac{-P_k}{(P_{k+1} - P_k)}$
    F_t is 0 and the the graph is undefined

0 case

By (7), when F_γ=0, t is at an extremum.

$$\sum_{k \geq 1} \frac{\gamma^{k-1} t^{k-1} d!}{(k + d - 1)!} P_{d+k}\left(\frac{\gamma t}{k + d} - 1\right) = 0 \tag{8}$$

If solutions exist, the arrival times display non-monotonicity.

Conclusion

Using our more robust form of the SIR Model and the use of meta-population models we can derive how a disease expands with the uneven flow of population growth and travel.

We found something rather counter intuitive, typically when the number of flights required to get from one node to another is higher the probability an infected person will arrive at the end location is strictly decreasing. For some graphs this is not the case. We found that for the following probability series, the derivative in (7) is undefined.

$$P_{k+1} = P_k\left(\frac{\gamma - 1}{\gamma}\right) \tag{9}$$

We found the probability matrix for a square and line graph of infinite nodes and concluded that for both graphs there is some set of initial and final nodes that imply a non-monotone decreasing function.

Future Work

In the future we can expand this work by attempting to derive the probability matrices for more complex graphs and perhaps a simple graph that describes the general layout of the airports around the world. This would provide us with a general means of describing the motion of a disease as it expands throughout the world.

References

[1] Switchover phenomenon induced by epidemic seeding on geometric networks [Ódor, G., Czifra, D., Komjáthy, J., Lovász, L., & Karsai, M. (2021). Switchover phenomenon induced by epidemic seeding on geometric networks. Proceedings of the National Academy of Sciences, 118(41).]

[2] Epidemic spreading on complex networks as front propagation into an unstable state [Armbruster, A., Holzer, M., Roselli, N., & Underwood, L. (2021). Epidemic spreading on complex networks as front propagation into an unstable state. arXiv preprint arXiv:2109.11985.]

[3] Population dispersal via diffusion-reaction equations [Kandler, A., & Unger, R. (2010). Population dispersal via diffusion-reaction equations.]

[4] "The Hidden Geometry of Complex, Network-Driven Contagion Phenomena". [Dirk Brockmann and Dirk Helbing. In: Science 342.6164 (2013), pp. 1337–1342. doi: 10.1126/science.1245200.]

[5] Mathew George (2022). B-A Scale-Free Network Generation and Visualization (https://www.mathworks.com/matlabcentral/fileexchange/11947-b-a-scale-free-network-generation-and-visualization), MATLAB Central File Exchange. Retrieved May 2, 2022.