Combinatorial Formulas for the Equivariant Cohomology of Peterson Varieties

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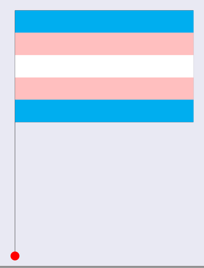
Complete Flag Variety

$$X = Fl(\mathbb{C}^n) = \{0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n | \dim(V_i) = i\}$$

Each point in X is a chain of vector spaces.



$$\mathbb{R}^0 \subseteq \mathbb{R}^1 \subseteq \mathbb{R}^2$$

Peterson Variety

The Peterson variety

$$Y \subseteq X$$

is the collection of flags satisfying

$$MV_i \subset V_{i+1} \quad 1 \leq i \leq n-1$$

where M is a principal nilpotent operator, i.e., a matrix with one Jordan block with 0s on the diagonal.

Schubert classes on X and on Y

Basis for $H_S^*(X)$: Schubert classes σ_v, indexed by elements of S_n.

Basis for $H_S^*(Y)$: Peterson classes, p_I each indexed by $I \subseteq [n-1] = \{1, 2, \cdots n-1\}$. Peterson classes are all images of specific Schubert classes under

$$\iota^* : H_S^*(X) \longrightarrow H_S(Y)$$

Combinatorics of Algebraic Varieties

Variety	Basis Classes	Index Set
$Fl(\mathbb{C}^n)$	Schubert classes (σ_w)	w$\in S_n$
Pet(n)	Peterson Schubert Classes p_A	$A \subseteq [n-1]$

Goal

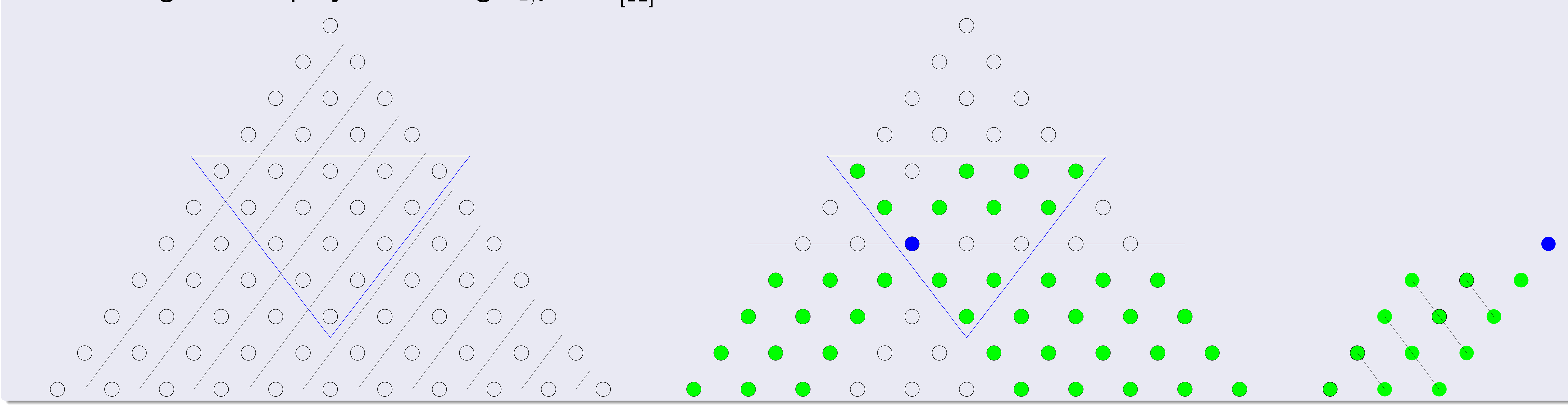
We express the restriction of transposition Schubert classes to the Peterson variety as a linear combination of Peterson classes:

$$\iota^*(\sigma_w) = \sum_{A \subset [n-1]} b_w^A p_A.$$

We want to find a *positive formula* for the coefficients b_w^A.

Combinatorial Triangles

These triangles exemplify restricting $\tau_{1,8}$ to $w_{[11]}$.



Main Tool: Localization

Let W_A be reduced-word representation for w_A of the following form: For each consecutive subset of A, without loss of generality $\{a, a+1, \cdots b\}$, we multiply $(s_a s_{a+1} \cdots s_{b-1})(s_a \cdots s_{b-2}) \cdots (s_a s_{a+1}) s_a$. Then

$$\iota^*(\sigma_u)|_{w_A} = \sum_{U \in \rho(u)} n_{W_A}(U) \left(\prod_{j \in U} (j - \mathcal{T}_A(j) + 1) \right)$$

where $\rho(u)$ is the set of reduced words of u, $n_{W_A}(U)$ is the number times the word U occurs as a subword of W_A, and $\mathcal{T}_A(j)$ is the smallest integer in the maximal consecutive subset of A containing j.

Subword Counts

$c_j = s_1 s_2 \cdots s_j$ is a subword of $w_{[m]}$ where $W_{[m]} = (s_1 s_2 \cdots s_m)(s_1 \cdots s_{m-1}) \cdots (s_1 s_2) s_1$ in $\binom{m}{j}$ different ways.

Theorem (Braid Cardinality Theorem)

Let

$$\mathrm{Br}_m(b_0; \tau_{ij}) = \sum_{a=j-b_0}^{m-j+2} \binom{a+b_0-i-1}{b_0-i} \binom{m+1-i-a}{b_0-i} \binom{a-1}{j-b_0-1} \binom{m-a-b_0+1}{j-b_0-1}$$

be the braid cardinality of τ_{ij} with braid index b_0. Then, we have that

$$\sum_{U \in R(\tau_{ih})} n_{W_{[m]}}(U) = \sum_{a=\max(2j-m,1)}^{j} \mathrm{Br}(b_0; \tau_{ij}).$$

Braid Cardinality Combinatorics

The braid cardinality of τ_{1j} for braid index b_0 can be simplified to

$$\mathrm{Br}_m(b_0; \tau_{1j}) = \binom{j-2}{b_0-1}^2 \binom{m+b_0-1}{2j-3} \quad (1)$$

Our Conjecture

Let τ_{ij} be the transposition of i and j, where $i < j$, and call $m \equiv j - i$ the magnitude of the transposition. We have that

$$\iota^*(\sigma_{\tau_{ij}}) = \sum_{k=0}^{m-1} \sum_{h=0}^{k} h! \binom{k}{h}^2 \binom{m-1}{k}^2 t^h p_{\{1+i+k-m, \cdots j+k-h-1\}} \quad (2)$$

excluding terms where $1 + i - k - m < 1$ or $j + k - h \geq n$.

Theorem ($i = 1$ case, proven by us)

Let τ_{ij} be the transposition of 1 and j. Then

$$\iota^*(\sigma_{\tau_{1j}}) = \sum_{h=0}^{j-2} h! \binom{j-2}{h}^2 t^h p_{[2j-h-3]} \quad (3)$$

Challenges to Proving Conjecture for τ_{ij}

The last challenge to proving our conjecture for arbitrary transpositions is using combinatorial identities to show equivalence of our conjecture and the sum of the braid counts derived from the visual triangle representation. One approach we've taken is employing the Egorychev method for deriving identities for sums of binomial coefficients.

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