Analyzing Monotonicity in the Linearized SIR Model

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Introduction

Figure: Airport traffic within the United States

- The spread of disease across far away regions via travel
- The more connected a population center is, the greater the chance of an out break occurring
- How can we model the flow of infectious groups between population centers to determine when an infected population center will infect another?

Meta-Population SIR model

S	How many people are susceptible
I	How many people are infected
R	How many people are removed
α	Infection rate
β	Removal rate
γ	Dispersion or "travel" rate
P	Probability matrix

Base SIR model

$$\begin{aligned}\frac{dS}{dt} &= -\alpha IS \\ \frac{dI}{dt} &= \alpha IS - \beta I \\ \frac{dR}{dt} &= \beta I\end{aligned}$$

$$\begin{aligned}\frac{dS_n}{dt} &= -\alpha I_n S_n + \gamma \sum_m P_{nm}(S_m - S_n) \\ \frac{dI_n}{dt} &= \alpha I_n S_n - \beta I_n + \gamma \sum_m P_{nm}(I_m - I_n) \\ \frac{dR_n}{dt} &= \beta I_n + \gamma \sum_m P_{nm}(R_m - R_n)\end{aligned}$$

S_n, I_n, and R_n are the susceptible, infected, and removed populations at a "city" n

Graph Theory

Going from Node 3 to Node 2:
Powers of the Weighted Adj. Matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^1 \quad (2,3) = \frac{1}{3}$$

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{4} & \frac{5}{12} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{4} & \frac{1}{6} & \frac{5}{12} \end{pmatrix}^2 \quad (2,3) = \frac{1}{6}$$

$$\begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{2}{9} & \frac{1}{6} & \frac{11}{36} & \frac{11}{36} \\ \frac{1}{12} & \frac{11}{24} & \frac{1}{6} & \frac{7}{24} \\ \frac{1}{12} & \frac{11}{24} & \frac{7}{24} & \frac{1}{6} \end{pmatrix}^3 \quad (2,3) = \frac{11}{36}$$

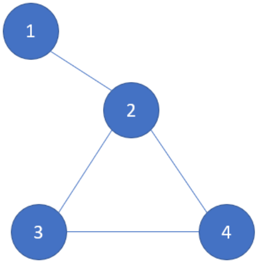


Figure: Meta-population model

Purpose

- γ represents the share of the population that are travelling at an instant in time
- One would expect that increasing γ, the travel rate, would lead to faster accumulation to our disease threshold from one population center to another.
- Some graphs do not show a monotone decreasing arrival time as a function of gamma
- Our purpose this semester was to determine a common property among graphs that do not display a monotone decreasing arrival time or determine in a graph between which nodes can we observe a non-monotone decreasing arrival time

Linearized SIR Model

It is known that a linearized form of the nonlinear system sufficiently estimates it $(S = 1 - s, I = i)$

$$\frac{di_n}{dt} = (\alpha - \beta)i_n + \gamma \sum_m P_{nm}(i_m - i_n) \tag{1}$$

Matrix Form

$$\frac{di_n}{dt} = (\alpha - \beta) i_n + \gamma \sum_m P_{nm} i_m - \gamma i_n \sum_m P_{nm}$$

$$= (\alpha - \beta) i_n + \gamma \sum_m P_{nm} i_m - \gamma i_n = (\alpha - \beta - \gamma) i_n + \gamma \sum_m P_{nm} i_m$$

Condense system of equations with matrices

$$\frac{d\mathbf{i}}{dt} = (\alpha - \beta - \gamma)\mathbf{i} + \gamma P \mathbf{i} = (\alpha - \beta)\mathbf{i} + \gamma (P - \mathbb{I})\mathbf{i}$$

Apply transformation $\{\tau = (\alpha - \beta)t, \tilde{\gamma} = \frac{\gamma}{\alpha - \beta}\}$ to focus on γ

$$(\alpha - \beta)\frac{d\mathbf{i}}{d\tau} = (\alpha - \beta)\mathbf{i} + (\alpha - \beta)\tilde{\gamma}(P - \mathbb{I})\mathbf{i}$$

$$\frac{d\mathbf{i}}{d\tau} = \mathbf{i} + \tilde{\gamma}(P - \mathbb{I})\mathbf{i} = (1 - \tilde{\gamma})\mathbf{i} + \tilde{\gamma} P \mathbf{i} = ((1 - \tilde{\gamma})\mathbb{I} + \tilde{\gamma} P)\mathbf{i}$$

Rename variables

$$\frac{d\mathbf{i}}{dt} = ((1 - \gamma)\mathbb{I} + \gamma P)\mathbf{i} \tag{2}$$

Explicit Solution

$$\mathbf{i}(t) = e^{((1-\gamma)\mathbb{I}+\gamma P)t}\mathbf{i}(0) = \left(e^{(1-\gamma)t} \sum_{k=0}^{\infty} \frac{t^k \gamma^k P^k}{k!} \right) \mathbf{i}(0)$$

We look at the particular case where at $t = 0$ node 1 has an infection level of 1, and all other nodes have 0 infection. So, $\mathbf{i}(0) = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}^\intercal$. Define $P_k := (P^k)_{n1}$ for a fixed node label n.

$$i_n(t) = F(\gamma, t) = e^{(1-\gamma)t} \sum_{k=0}^{\infty} \frac{t^k \gamma^k P_k}{k!} = 1 \tag{3}$$

Total Differential Equation

$$
\begin{aligned}
F(\gamma, t) = i_n(t) &= 1 \\
F_\gamma + F_t \frac{dt}{d\gamma} = \frac{dF}{d\gamma} = \frac{d}{d\gamma} 1 &= 0 \\
\frac{dt}{d\gamma} = -\frac{F_\gamma}{F_t}
\end{aligned}
$$

when this derivative is equal to zero or undefined we get the non-monotonicity that we are looking for.

$$
\begin{aligned}
F_\gamma &= e^{(1-\gamma)t} \sum_{k=0}^{\infty} \frac{t^{k+1}\gamma^k}{k!}(P_{k+1} - P_k) \\
F_t &= e^{(1-\gamma)t} \sum_{k=0}^{\infty} \frac{t^k \gamma^k}{k!}(P_k + \gamma(P_{k+1} - P_k))
\end{aligned}
$$

Particular Cases

- When P_k is non-constant (no disconnected graphs) and increasing, the partial derivatives are positive, so $\frac{dt}{d\gamma}$ is negative and thus the arrival times are decreasing.
- When $F_\gamma = 0$, the derivative $\frac{dt}{d\gamma} = 0$. Proofs that $F_\gamma = 0$ somewhere would provide explicit conditions for non-monotonicity arrival times. (Arrival times increasing then decreasing after passing a certain dispersion rate threshold, or vice versa).
- When $F_t = 0$, the derivative is undefined, resulting in a vertical asymptote.

The Undefined Case

When

$$F_t = e^{(1-\gamma)t} \sum_{k=0}^{\infty} \frac{t^k \gamma^k}{k!} (P_k + \gamma(P_{k+1} - P_k)) = 0$$

the total derivative is undefined and the arrival time approaches infinity. This will occur when the following geometric relation is upheld on an infinite graph

$$P_{k+1} = P_k(\frac{\gamma - 1}{\gamma})$$

From this geometric sequence we can determine that the undefined case will occur when $\gamma > 1$

when $\gamma < 1$

$$\frac{\gamma - 1}{\gamma} < 0$$

this would imply a negative probability and is thus invalid

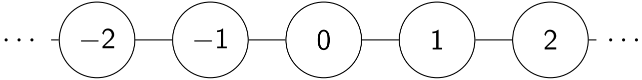
when $\gamma = 1$

$$\frac{\gamma - 1}{\gamma} = 0$$

this would imply a 0 probability and thus a disconnected graph and is thus invalid

Line Lattices

On a 1-dimensional lattice, the walks of length k are isomorphic to the sequences of length k whose terms are either 1 or -1.

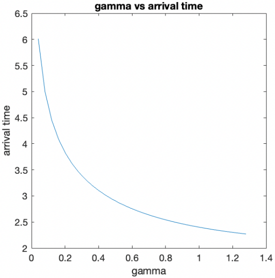


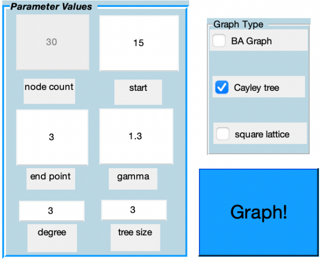
Thus asking how many walks of length k are there that end at a node exactly d edges away is the same as asking how many sign sequences of length k sum to d. If the number of -1's in the sequence is m, then the sequence sums to d and $(k-m)-m=k-2m$. There are $\binom{k}{m}=\binom{k}{\frac{k-d}{2}}$ unique sequences.

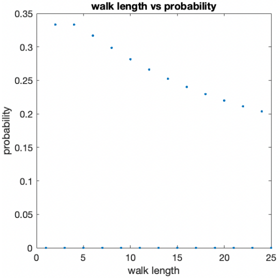
$$P_k = \frac{\binom{k}{\frac{k-d}{2}}}{2^k} \text{ if } k-d \text{ even}$$

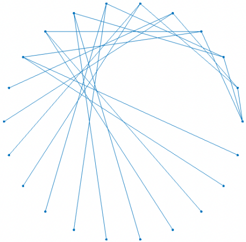
Probability increases to a peak then decreases, approaching 0. Non-monotone arrival times.

Some Really Cool Graphs

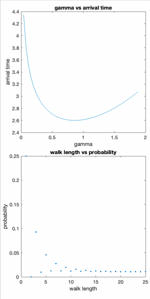


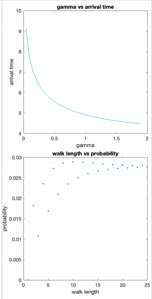




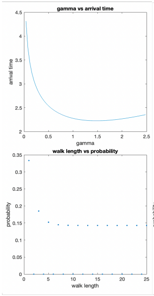


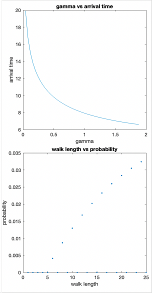
Even More Really Cool Graphs



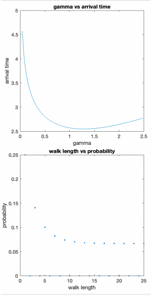


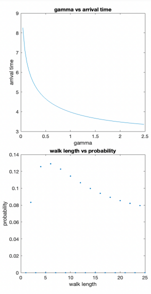
Barabási–Albert





Caylee Tree





Square Lattice

Conclusion

We analyzed the relationship between dispersion and arrival times and determined several conditions for specific effects on the arrival times in terms of probability sequences. Increasing probability sequences guarantee decreasing arrival times, and some special cases guarantee non-monotonic arrival times. We also used limiting models in the form of infinite lattice and tree graphs to approach simulating large networks.

Future Research

- study more strict requirements for monotone and non monotone graphs
- develop probability equations for the square lattice and the Caylee tree
- Study networks of the general layout of the airports around the world to determine the validity of our model in the real world

The End

Thanks for Watching!

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