Combinatorics of Cohomology Rings of the Peterson Variety: Transpositions

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Introduction and Motivation

- Let X and Y be two rings with additional $\mathbb{C}[t]$-module structure.
- X has module basis $\{\sigma_w | w \in S_n\}$ where S_n is the permutation group on n symbols.
- Y has module basis $\{p_A | A \subseteq \{1, 2, \cdots, n-1\}\}$.
- We have a surjective ring map $\iota^* : X \to Y$.
- Let $v_A = \Pi_{j \in A} s_j$ ordered with lower js to the left, where $s_j = (j, j+1)$. Then
$$p_A = \iota^* \sigma_{v_A}.$$
- Question: What is $\iota^* \sigma_w$ in terms of the $\{p_A\}$ for $w \neq v_A$?

Transposition Conjecture

Let τ_{ij} be the transposition of i, j, where $i < j$. Let $m := j - i$. Then

$$\iota^*(\sigma_{\tau_{ij}}) = \sum_{k=0}^{m-1} \sum_{h=0}^{k} h! \binom{k}{h}^2 \binom{m-1}{k}^2 t^h p_{[1+i+k-m, j+k-h-1]}$$

excluding any terms where $1 + i - k - m < 1$ or $j + k - h \geq n$, where $[a, b] = \{a, a+1, \cdots b-1, b\}$.

A Simpler Case Proven

Let τ_{1j} be the transposition of $1, j$, where $j > 1$. Then

$$\iota^*(\sigma_{\tau_{1j}}) = \sum_{h=0}^{j-2} h! \binom{j-2}{h}^2 t^h p_{[2j-h-3]},$$

where $[2j - h - 3] = \{1, 2, \cdots 2j - h - 3\}$.

Reduced words for τ_{1j}

The shortest possible strings of $s_i = \tau_{i,i+1}$ that multiply out to τ_{1j}. They have length $2j-3$. They are related by commutation (i.e. $s_1 s_3 = s_3 s_1$) and braid moves ($s_1 s_2 s_1 = s_2 s_1 s_2$):

$$\begin{aligned}
\tau_{1j} &= s_1 s_2 s_3 s_4 \cdots s_{j-3} s_{j-2} s_{j-1} s_{j-2} s_{j-3} \cdots s_2 s_1 \\
&= s_1 s_2 s_3 s_4 \cdots s_{j-3} s_{j-1} s_{j-2} s_{j-1} s_{j-3} \cdots s_2 s_1 \\
&= s_{j-1} s_1 s_2 s_3 s_4 \cdots s_{j-3} s_{j-2} s_{j-1} s_{j-3} \cdots s_2 s_1 \\
&\vdots \\
&= s_{j-1} s_{j-2} s_{j-3} \cdots s_3 s_2 s_1 s_2 s_3 \cdots s_{j-3} s_{j-2} s_{j-1}
\end{aligned}$$

Localization

We used a technique called localization and reduced the proof to calculating this:

$$\sum_{U \text{ red. for } \tau_{1j}} n_{W_{[m]}}(U)/b(U)$$

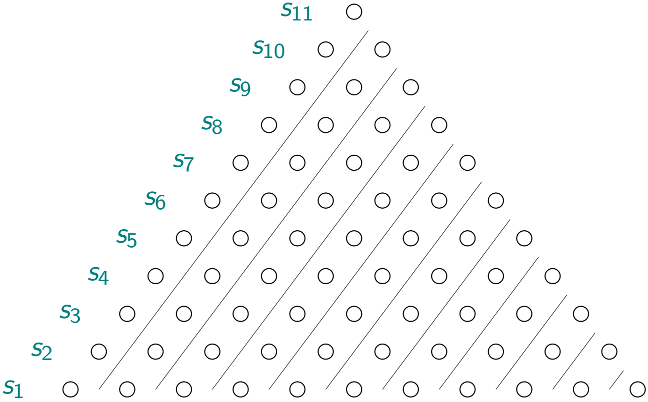
where

$$W_{[m]} = (s_1 s_2 \cdots s_{m-1})(s_1 s_2 \cdots s_{m-2}) \cdots (s_1 s_2) s_1,$$

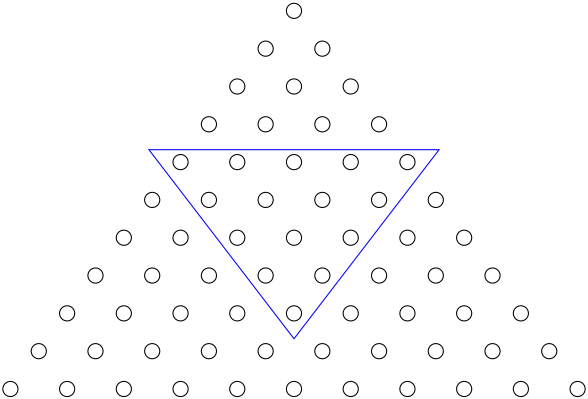
we are summing over reduced words U for τ_{1j}, $n_{W_{[m]}}(U)$ is the number of ways the reduced word U fits into $W_{[m]}$, and $b(U)$ is the word's braid index.

Triangle Representation of a Long Word

The following triangle represents $W_{[11]}$. A dot in row i represents simple reflection s_i. The j^{th} diagonal from the leftmost diagonal represents $s_1 s_2 \cdots s_{11-j}$.

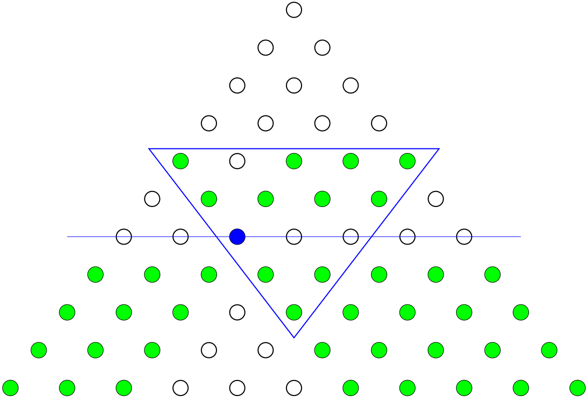


The blue triangle contains possible indices for the simple reflection with a unique index.



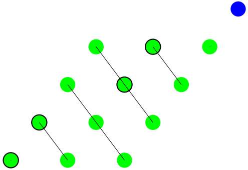
Triangle: Possible Ascending/Descending Subwords

Choosing one braid node, we get 4 parallelograms of green dots that contain possible ascending and descending subwords.



Triangle: Choosing Diagonals

In each parallelogram, we can choose a number of diagonals equal to the dot height from the number of choices equal to the dot height plus the dot length of the base minus 1.



Counting Elements of a Braid Index

By looking at all possible braid node choices on row $b_0 \leq m$ on the triangle representation of $W_{[m]}$, we found that there are

$$\mathrm{Br}_m(b_0; \tau_{1j}) = \sum_{a=j-b_0}^{m-j+2} \binom{a+b_0-2}{b_0-1}\binom{m-a}{b_0-1} \times \binom{a-1}{j-b_0-1}\binom{m-a-b_0+1}{j-b_0-1}. \quad (1)$$

ways to write expressions of τ_{1j} with braid index a as subwords of $W_{[m]}$. Then

$$\sum_U n_{W_{[m]}}(U)/b(u) = \sum_{b=1}^{j-1} Br_m(b; \tau_{1j})/b.$$

Simplifying $\mathrm{Br}_m(a; \tau_{1j})$

We use the identity

$$\binom{n}{h}\binom{n-h}{k} = \binom{n}{h+k}\binom{h+k}{h}$$

on the first and third, as well as second and fourth pairs in the summands of Eq. (1) to get

$$\mathrm{Br}_m(b_0; \tau_{1j}) = \binom{j-2}{b_0-1}^2 \left(\sum_{a=j-b_0}^{m-j+2} \binom{a+b_0-2}{j-2}\binom{m-a}{j-2} \right).$$

By reindexing with $c = a + b_0 - 2$, we obtain

$$\mathrm{Br}_m(b_0; \tau_{1j}) = \binom{j-2}{b_0-1}^2 \left(\sum_{c=j-2}^{m-j+b_0} \binom{c}{j-2}\binom{m-c+b_0-2}{j-2} \right) \quad (2)$$

Simplifying $\mathrm{Br}_m(a; \tau_{1j})$

We can use that

$$\sum_{c=\kappa}^{\mu-\kappa} \binom{c}{\kappa} \binom{\mu - c}{\kappa} = \binom{\mu + 1}{2\kappa + 1}$$

to get that

$$\mathrm{Br}_m(b_0; \tau_{1j}) = \binom{j-2}{b_0 - 1}^2 \binom{m + b_0 - 1}{2j - 3}, \tag{3}$$

Our Theorem Rephrased

If we prove that

$$\frac{1}{m}\binom{m}{j-1}^2 = \sum_{k=1}^{j-1} \frac{1}{k}\binom{j-2}{k-1}^2 \binom{m+k-1}{2j-3}$$

we're done. This is equal to the assertion that

$$\binom{m}{j-1}\binom{m-1}{j-2} = \sum_{k=1}^{j-1} \binom{j-1}{k}\binom{j-2}{k-1}\binom{m+k-1}{2j-3}.$$

Closing Out The Proof

There is an identity that

$$\sum_{i=0}^{\min(a,b)} \binom{x+y+i}{x+y}\binom{y}{y+i-a}\binom{x}{x+i-b} = \binom{x+a}{x}\binom{y+b}{y}.$$

We can re-index to make that prove that

$$\binom{m}{j-1}\binom{m-1}{j-2} = \sum_{k=1}^{j-1} \binom{j-1}{k}\binom{j-2}{k-1}\binom{m+k-1}{2j-3},$$

so we're done.

Next Steps

- We want to prove the general case with τ_{ij}.
- We want to be able to restrict transpositions to long words that don't necessarily start at 1.
- Afterwards, we'd want to be able to compute the pullback of these transpositions.
- We want to publish our results.