Mathematics of Floating 3D Printed Objects

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Mathematical Models of Floating Objects

Definition (Center of Gravity)

If the mass distribution is given by a continuous density function of $\rho(x, y, z)$ within a domain Ω, then the center of gravity can be obtained by

$$(G_x, G_y, G_z) = \frac{1}{M_{\mathrm{obj}}} \iiint_\Omega (x, y, z)\, \rho(x, y, z)\; dV \ ,$$

For a cross sectional object with length L and a uniform density the center of gravity is given by

$$\vec{G} = (G_x, G_y) = \frac{L}{M_{\mathrm{obj}}} \iint_\Omega (x, y) \rho(x, y)\; dA$$

with $G_z = L/2$

Area of a General Polygon

Lemma

Let

$$\{(x_1, y_1), \ldots, (x_N, y_N), (x_1, y_1)\}$$

be the vertices of the cross section polygon, oriented counterclockwise. Then the mass of the object is

$$M_{\mathrm{obj}} = \rho L A \ ,$$

where the area A of the polygon of cross section is given by

$$A = \frac{1}{2} \sum_{k=1}^{N} (x_k + x_{k+1})(y_{k+1} - y_k)$$

Lemma

$(x, y) = (1-t)(x_k, y_k) + t(x_{k+1}, y_{k+1})$ $dy = (y_{k+1} - y_k)dt.$

$$\begin{aligned}
\int_0^1 x\, dy &= \int_0^1 (x_k + t(x_{k+1} - x_k))(y_{k+1} - y_k)\, dt \\
&= x_k(y_{k+1} - y_k) + \frac{1}{2}(x_{k+1} - x_k)(y_{k+1} - y_k) \\
&= \frac{1}{2}(x_{k+1} + x_k)(y_{k+1} - y_k)\,.
\end{aligned}$$

Now we use this to calculate the full area:

$$A = \iint_{\Omega} dA = \oint_{d\Omega} x\, dy = \frac{1}{2}\sum_{k=1}^{N}(x_{k+1} + x_k)(y_{k+1} - y_k)\,.$$

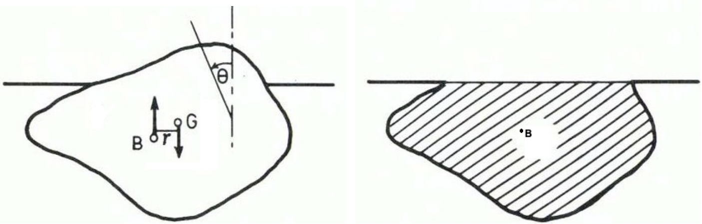
General Polygon $\vec{G}$

Lemma

By the shoelace formula, and the center of gravity $\vec{G} = (G_x, G_y)$ is given by

$$\begin{aligned} G_x &= \frac{1}{6A}\sum_{k=1}^{N}(x_k^2 + x_k x_{k+1} + x_{k+1}^2)(y_{k+1} - y_k) \\ G_y &= \frac{1}{6A}\sum_{k=1}^{N} -(y_k^2 + y_k y_{k+1} + y_{k+1}^2)(x_{k+1} - x_k)\,. \end{aligned} \tag{1}$$

Center of Buoyancy, $\vec{B}$



Definition

The centroid of the displaced fluid

$$(B_x, B_y, B_z) = \frac{1}{V_{\text{sub}}} \iiint_{\Omega_{\text{sub}}} (x, y, z) dV$$

Archimedes' Principle

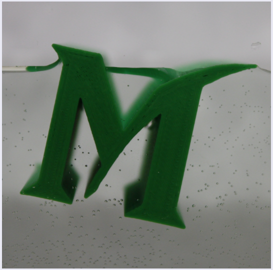
Definition

The upward buoyant force exerted on an object wholly or partially submerged is equal to the weight of the displaced fluid

$$M_{obj}g = \rho_{fluid} V_{sub} g$$

$$\frac{V_{sub}}{V_{obj}} = \frac{\rho_{obj}}{\rho_{fluid}} = R$$

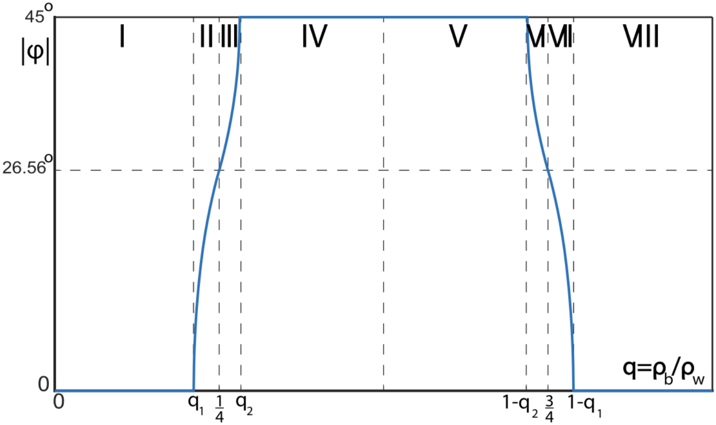
For the floating Mason M print with $\rho \approx 0.8$



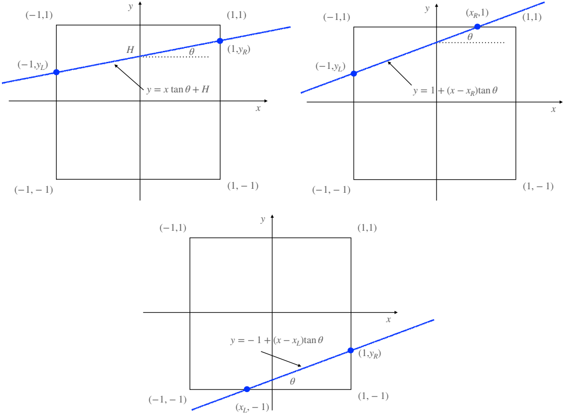
Algorithm: Compute Potential Energy Landscapes

Code developed to produce PE landscape plots and test stability uses the algorithm:

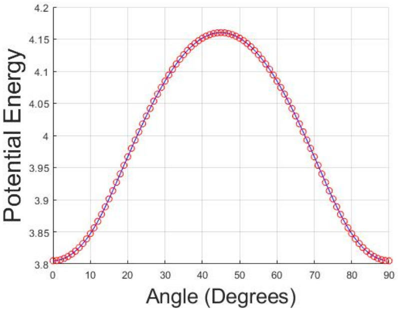
- Given an arbitrary shape or a set of boundary points
- Compute $\vec{G}$ (center of mass)
- For $\theta \in [0, 2\pi]$ (orientation of object)
  - Identify water line consistent with Archimedes'
  - Compute $\vec{B}(\theta)$ (center of buoyancy)
  - Potential Energy
$$U(\theta) \sim \hat{n}(\theta) \cdot (\vec{G} - \vec{B}(\theta))$$

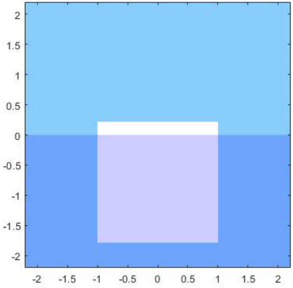

45°
|φ|
26.56°
0
I
II
III
IV
V
VI
VII
VIII
q=ρ_b/ρ_w
0
q_1
1/4
q_2
1-q_2
3/4
1-q_1

Square Cross Section Revisited



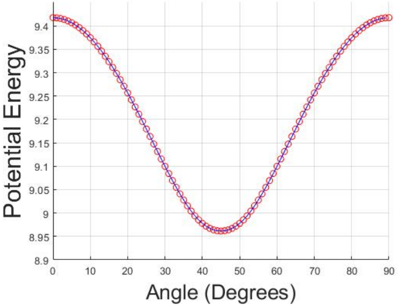
Potential Energy Landscape for Square when R=0.8912

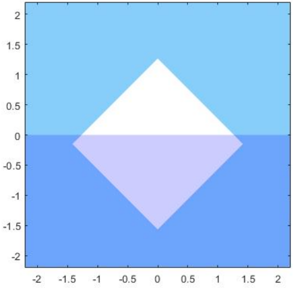




- Numerical solution and theory agree
- Stable equilibrium angle$= 0, 90, 180, 270$, or equivalent

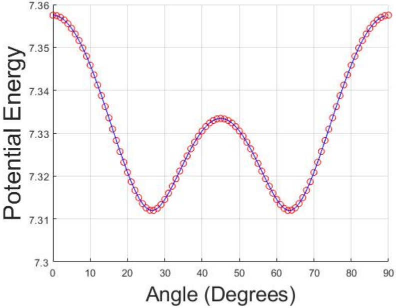
Potential Energy Landscape for Square when R=0.6

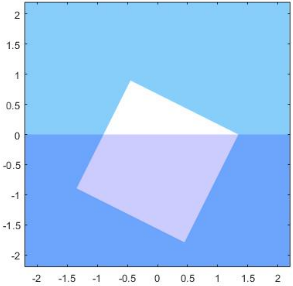




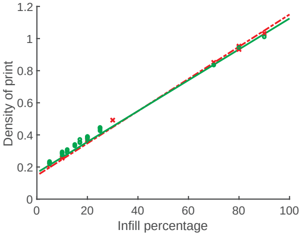
- Numerical solution and theory agree
- Stable equilibrium angle$= 45, 135$, or equivalent

Potential Energy Landscape for Square when R=0.75

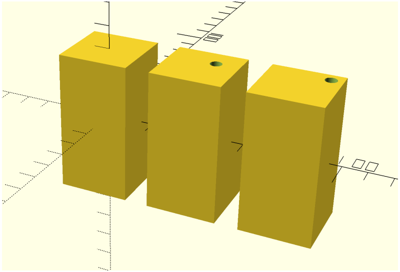




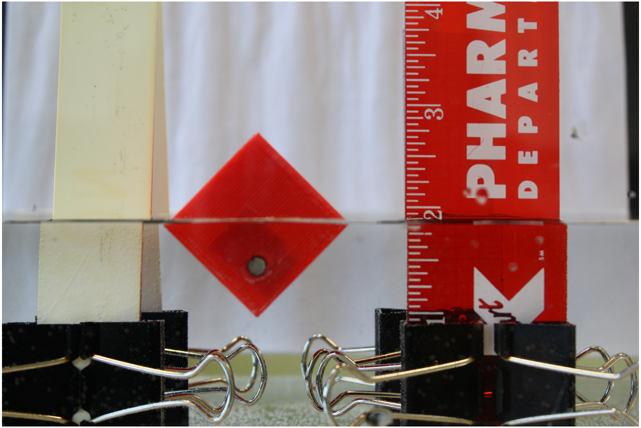
- Numerical solution and theory agree
- Stable equilibrium angle= $26.56, 63.44$,or equivalent



The predicted values without hole (dashed red) and with hole (solid green), graphed along with the measured values without hole (red x) and with hole (green o).

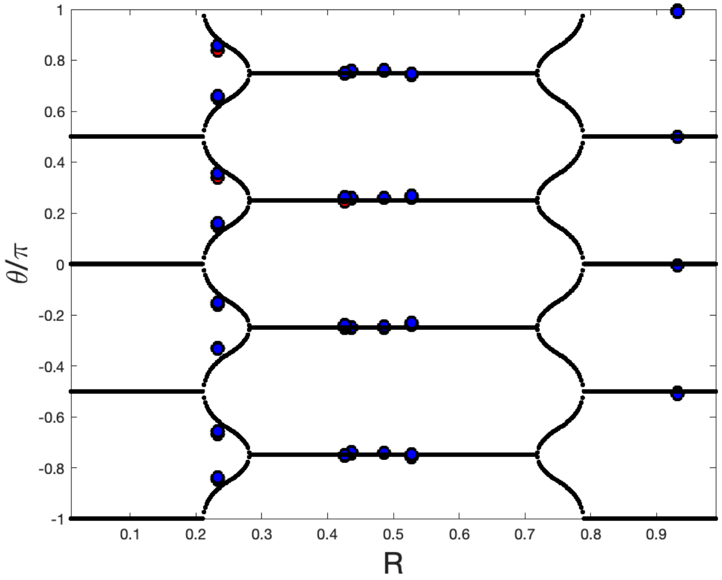


Experimental Setup

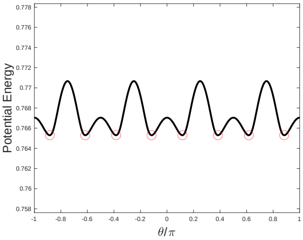


The experimental set up to measure angles associated with stable floating orientations.

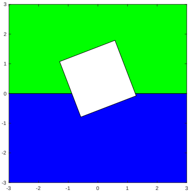
Stable orientations versus density ratio for $\vec{G} = (0, 0)$

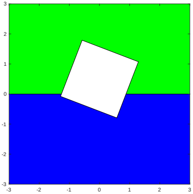


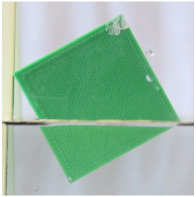
Potential Energy for Square Cross Section $R = 0.23296$

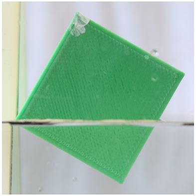


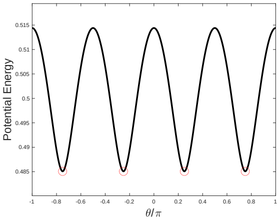
The potential energy for the square with $\vec{G} = (0, 0)$ and $R = 0.23296$. There are eight stable equilibria. Two stable floating configuration corresponding to the orientations closest to $\theta = 0$.



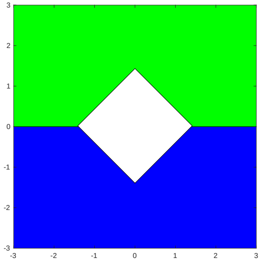


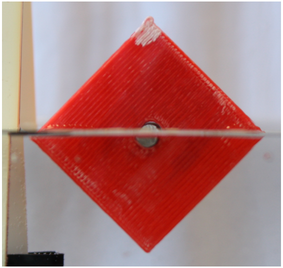




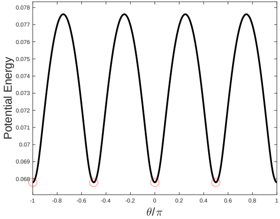


The potential energy for the square with $\vec{G} = (0, 0)$ and $R = 0.4856$. There are four stable equilibria corresponding to $\theta = \frac{\pi}{4} \pm \frac{\pi}{2} n$ for integer n.

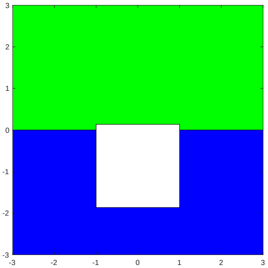


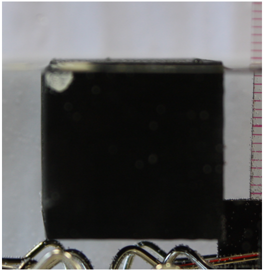


Potential Energy for Square Cross Section $R = 0.9322$

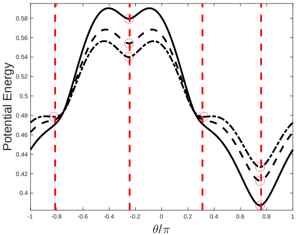


The potential energy for the square with $\vec{G} = (0,0)$ and $R = 0.9322$. There are four stable equilibria corresponding to $\theta = \pm\frac{\pi}{2}n$ for integer n.

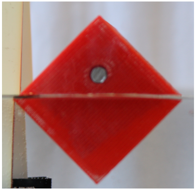
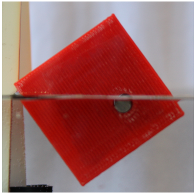
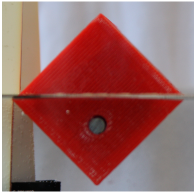
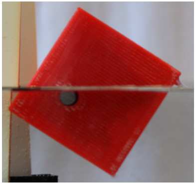




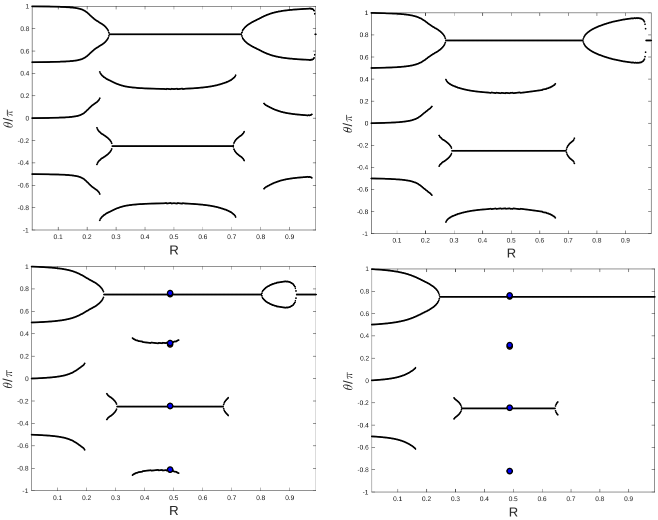
Breaking Symmetry

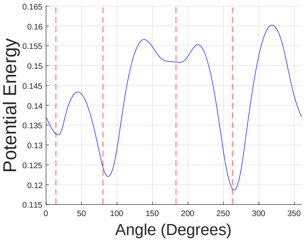


Our theory predicts only two stable orientations, but we observe four experimental stable orientations. The plot shows two other potential energy landscapes for nearby values of $\vec{G}$ indicating the presence of nearby stable states.

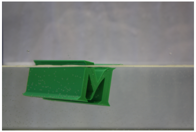





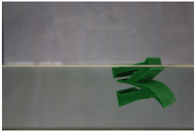
Equilibrium Angles vs Density Ratio for Off Center Squares

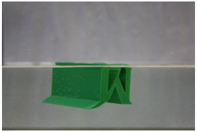


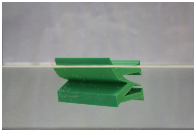


The potential energy of the Mason M. Each vertical red dotted line corresponds to an experimentally found stable orientation. The local minima of the graph define stable orientations theoretically calculated using our code.









References

Daniel M. Anderson, Brandon G. Barreto-Rosa, Joshua D. Calvano, Lujain Nsair, and Evelyn Sander "Mathematics of Floating 3D Printed Objects." submitted for publication 2022.

Acknowledgements

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