Combinatorics of Cohomology Rings of Peterson Varieties

Dr. Brent Gorbutt, Martha Hartt, Ryan Simmons, Ziqi Zhan, Aidan Donahue

Mason Experimental Geometry Lab

George Mason University

May 6, 2022

Introduction

Our project looks for the relationships between binomial and multinomial coefficients in equations such as the combinatorial formula found by Goldin and Gorbutt that generalizes Vandermonde's Identity. We attempt to expand on Goldin/Gorbutt's work by finding more identities for binomial and multinomial coefficients in equations. To do this, we use a method called bike lock moves on sequences which are counted by the binomial and multinomial coefficients.

Theorem (Vandermonde's Identity)

Given a group of m objects of one type, and n of another, there are

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

ways to choose r objects from the group.

Theorem 9 (Goldin, Gorbutt)

Let $m, n, w, x, y, z \in$ such that $w + x = y + z$ and $m, n \geq 0$. Then

$$\binom{x+m}{w}\binom{y+m}{x}\binom{w+n}{y}\binom{z+n}{z} = \sum_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \binom{x+i+n}{w+i+j}\binom{w+m+j}{i, j, m-i, x-i-j, z-x+j, y-x+i}$$

Six Binomial Coefficients

While we do not believe a similar identity can be proven using bike lock moves for three or five binomial coefficients, we made progress in identifying the right hand side of an identity for

$$\binom{x_1+y_1}{x_1}\binom{x_1+y_6}{x_2}\binom{y_1+x_2}{x_3}\binom{y_6+x_1-x_2+x_3}{x_4}\times \binom{y_1+x_2-x_3+x_4}{x_5}\binom{x_6+y_6}{x_6}$$

where $x_1 + x_3 + x_5 = x_2 + x_4 + x_6$.

A Bijection of Sets

- Idea: Construct $\mathcal{V}$ and $\mathcal{S}$, such that $|\mathcal{S}| =$ RHS and $|\mathcal{V}| =$ LHS.
- Use the method of bike lock moves to find bijection between $\mathcal{V}$ and $\mathcal{S}$.
- We wish to apply the same process to find additional identities.

Table of Values

We used the following table to construct our matrices:

	1's	0's	$*$'s	sum
V_1	x_1	y_1	y_6	n
V_2	x_2	y_2	y_1	n
V_3	x_3	y_3	y_2	n
V_4	x_4	y_4	y_3	n
V_5	x_5	y_5	y_4	n
V_6	x_5	y_6	y_5	n

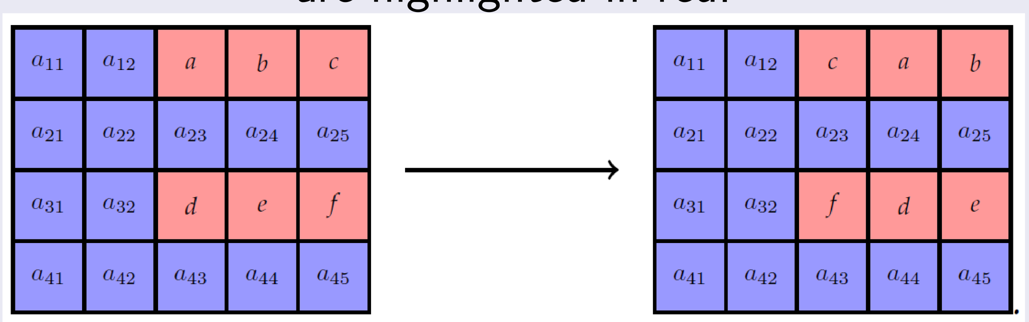
Definition (Bike Lock Move)[1]

For each k with $1 \leq k \leq c$, a bike love move BL_k on a set of matrices M_c with $c > 0$ columns is a map $M_c \to M_c$ such that, for all $M \in M_c$,

1. $BL_k(M)$ is identical to M except in a specified subset of rows $R_{BL_k(M)}$.
2. $BL_k(M)$ cyclically permutes the entries in row $\ell \in R_{BL_k(M)}$ as follows:
   - An entry in column $m < k$ is fixed.
   - An entry in column m with $k \leq m < c$ of M sent to column $m+1$ in the same row.
   - If $m = c$, the entry is sent to the kth column of the same row.

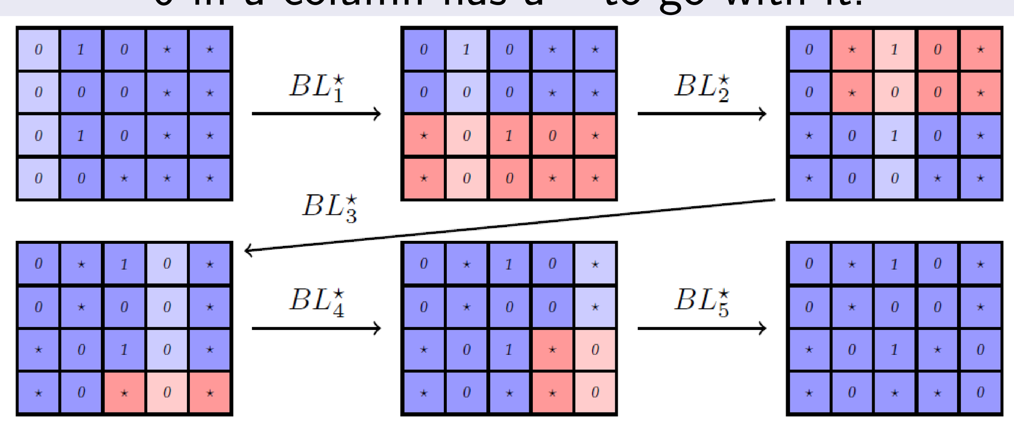
Bike Lock Move Example

A bike lock move BL_3 on a 4 x 5 matrix M with $R_{BL_3}(M) = \{1, 3\}$; can be seen as follows. Impacted entries are highlighted in red.



Aligning Matrix Entries

We use a series of bike lock moves on a matrix such that each 0 in a column has a * to go with it.



Example of Aligning an Entire Matrix

1	1	1	1	0	*	*	*	*	*	*
1	1	1	1	1	1	1	1	0	0	*
1	1	1	1	1	1	0	0	0	*	*
1	1	1	1	0	0	0	0	*	*	*
1	1	0	0	0	0	0	*	*	*	*
0	0	0	0	0	0	*	*	*	*	*

*	*	1	1	*	*	1	1	*	*	0
1	1	1	1	1	1	1	1	0	0	*
1	1	1	1	1	1	0	0	*	*	0
1	1	1	1	0	0	*	*	0	0	*
1	1	0	0	*	*	0	0	*	*	0
0	0	*	*	0	0	*	*	0	0	*

Possible Columns

$$c_1 = \begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} \quad c_2 = \begin{bmatrix}0\\ *\\1\\1\\1\\1\end{bmatrix} \quad c_3 = \begin{bmatrix}1\\0\\ *\\1\\1\\1\end{bmatrix} \quad c_4 = \begin{bmatrix}1\\1\\0\\ *\\1\\1\end{bmatrix} \quad c_5 = \begin{bmatrix}1\\1\\1\\0\\ *\\1\end{bmatrix} \quad c_6 = \begin{bmatrix}1\\1\\1\\1\\0\\ *\end{bmatrix} \quad c_7 = \begin{bmatrix}*\\1\\1\\1\\1\\0\end{bmatrix} \quad c_8 = \begin{bmatrix}0\\ *\\0\\ *\\1\\1\end{bmatrix} \quad c_9 = \begin{bmatrix}0\\ *\\1\\0\\ *\\1\end{bmatrix}$$

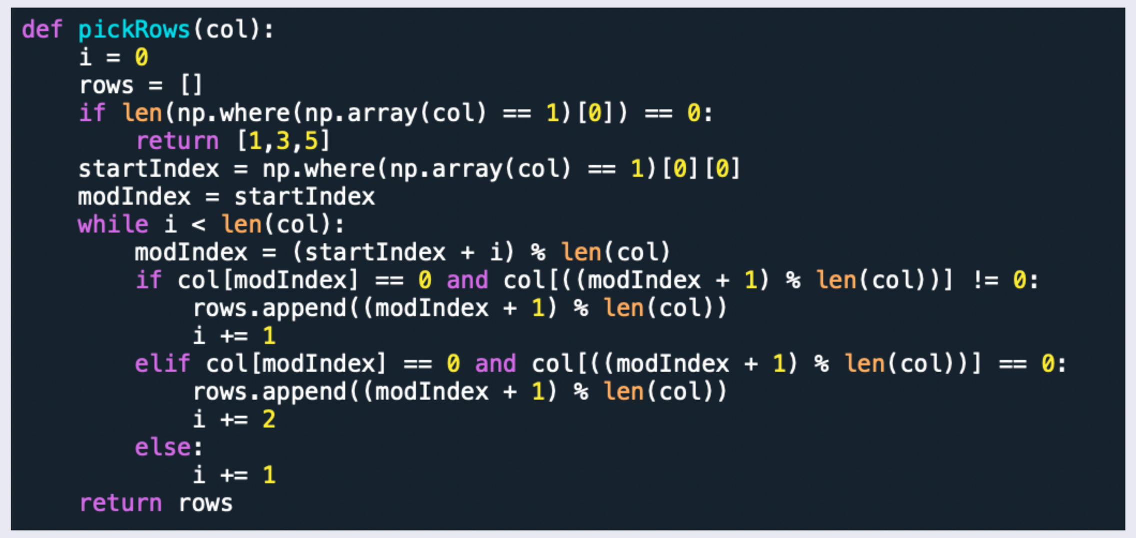
$$c_{10} = \begin{bmatrix}0\\ *\\1\\1\\0\\ *\end{bmatrix} \quad c_{11} = \begin{bmatrix}1\\0\\ *\\0\\ *\\1\end{bmatrix} \quad c_{12} = \begin{bmatrix}1\\0\\ *\\1\\0\\ *\end{bmatrix} \quad c_{13} = \begin{bmatrix}*\\0\\ *\\1\\1\\0\end{bmatrix} \quad c_{14} = \begin{bmatrix}1\\1\\0\\ *\\0\\ *\end{bmatrix} \quad c_{15} = \begin{bmatrix}*\\1\\0\\ *\\1\\0\end{bmatrix} \quad c_{16} = \begin{bmatrix}*\\1\\1\\0\\ *\\0\end{bmatrix} \quad c_{17} = \begin{bmatrix}0\\ *\\0\\ *\\0\\ *\end{bmatrix} \quad c_{18} = \begin{bmatrix}*\\0\\ *\\0\\ *\\0\end{bmatrix}$$

System of Equations

Now that we know all possible columns, we can create the following system of equations where C_i is the number of columns of type c_i:

$$\begin{aligned}
C_1 - C_8 - C_9 - C_{10} - C_{11} - C_{12} - C_{13} - C_{14} - C_{15} - C_{16} - 2C_{17} - 2C_{18} &= -x_1 + x_3 + x_5 - 2y_1 - 2y_6\\
C_2 + C_8 + C_9 + C_{10} + C_{12} &= y_1\\
C_3 + C_{11} + C_{12} + C_{13} + C_{18} &= x_1 - x_2 + y_6\\
C_4 + C_8 + C_{14} + C_{15} + C_{17} &= x_2 - x_3 + y_1\\
C_5 + C_9 + C_{11} + C_{16} + C_{18} &= x_1 - x_2 + x_3 - x_4 + y_6\\
C_6 + C_{10} + C_{12} + C_{14} + C_{17} &= x_2 - x_3 + x_4 - x_5 + y_1\\
C_7 + C_{13} + C_{15} + C_{16} + C_{18} &= y_6
\end{aligned}$$

Code Screenshot



```python
def pickRows(col):
    i = 0
    rows = []
    if len(np.where(np.array(col) == 1)[0]) == 0:
        return [1,3,5]
    startIndex = np.where(np.array(col) == 1)[0][0]
    modIndex = startIndex
    while i < len(col):
        modIndex = (startIndex + i) % len(col)
        if col[modIndex] == 0 and col[((modIndex + 1) % len(col))] != 0:
            rows.append((modIndex + 1) % len(col))
            i += 1
        elif col[modIndex] == 0 and col[((modIndex + 1) % len(col))] == 0:
            rows.append((modIndex + 1) % len(col))
            i += 2
        else:
            i += 1
    return rows
```

Given any unaligned column, the function above will tell us the rows we need to perform bike lock moves on. We conjecture that this function will allow us to align any matrix when the number of rows is even.

Progress

- Finished automation of bike lock moves to find the right hand side of an identity for six binomial coefficients.
- Found 18 possible aligned columns for the six case which we can use to find $\mathcal{S}$.
- Conjecture that more identities exist when you have an even number of binomial coefficients.
- Suspect no identity can be proven using bike lock moves when you have an odd number of binomial coefficients.

Future Explorations

- Identify the right hand side of an identity for six binomial coefficients.
- Prove our conjectures about cases when you have other numbers of binomial coefficients.
- Prove the code we used to automate the bike lock moves for six binomial coefficients works more generally.

Acknowledgements

We would like to thank our professor advisor, Dr. Gorbutt, and our graduate student mentor, Martha Hartt, for the guidance they provided this semester with our research.

References

1 Goldin, R., Gorbutt, B. (2022). A positive formula for type A Peterson Schubert calculus. ArXiv:2004.05959 [Math]. https://arxiv.org/pdf/2004.05959

2 Székely, L. (1985). Common Origin of Cubic Binomial Identities; A Generalization of Surányi's Proof on Le Jen Shoo's Formula. J. Comb. Theory Ser. A. **40**(1), 171-174