VERTEX OPERATOR ALGEBRAS:
FINITE-DIMENSIONAL CASES AND CONFORMAL BLOCKS

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Dedication

I dedicate this thesis to my siblings James, Katherine, and John.

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Table of Contents

Abstract

VERTEX OPERATOR ALGEBRAS: FINITE-DIMENSIONAL CASES AND CONFORMAL BLOCKS

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Vertex operator algebras are algebraic objects analogous to both commutative associative algebras with identity and Lie algebras. They provide a way of rigorously constructing a particular family of quantum field theories called rational conformal field theories. In this thesis we construct the simplest class of examples of vertex operator algebras, namely the finite dimensional ones, and prove basic results on modules of these vertex operator algebras and spaces of conformal blocks associated to smooth projective curves. We also construct the vertex operator algebra associated with the $\mathfrak{sl}_2(\mathbb{C})$ WZW model in the non-critical case. When combined with the FRS theorem for rational conformal field theories, vertex operator algebra theory can be used to rigorously construct one of the simplest examples of holographic duality: the Chern-Simons-WZW model correspondence.

Chapter 1: An introduction to notation and formal series

In this chapter we introduce notation that will be used throughout this text as well as definitions related to formal series. The chapter is concluded with proofs of useful formulae involving the formal delta function which will be used to show the simplest examples of vertex operator algebras satisfy the Jacobi identity.

1.1 Vector spaces of formal series and conventions

Note that in this thesis after Chapter 2, I am only considering vertex operator algebras defined over the base field $k = \mathbb{C}$. Other fields can be used in the definition of vertex operator algebra since the notion of summability of endomorphisms of a vector space is sufficiently restrictive, so the results on finite dimensional vertex operator algebras and modules still apply. However, in Chapter 3 the definition of conformal blocks only applies to the complex case, although there should be a way to generalize the definition by using Kahler differentials and taking the cohomology to have coefficients in k. Assume all vector spaces are complex unless stated otherwise. $\mathbb{N}$ is here defined as the nonnegative integers.

We define the following vector spaces associated with a vector space V.

$$V[[x,x^{-1}]] \equiv \{\sum_{n\in\mathbb{Z}} v_n x^n | v_n \in V\} \tag{1.1}$$

$$V[x] \equiv \{\sum_{n\in\mathbb{N}} v_n x^n | v_n \in V, \text{ all but finitely many } v_n = 0\} \tag{1.2}$$

$$V[x,x^{-1}] \equiv \{\sum_{n\in\mathbb{Z}} v_n x^n | v_n \in V, \text{ all but finitely many } v_n = 0\} \tag{1.3}$$

$$V[[x]] \equiv \{\sum_{n\in\mathbb{N}} v_n x^n | v_n \in V\} \tag{1.4}$$

$$V((x)) \equiv \{\sum_{n\in\mathbb{Z}} v_n x^n | v_n \in V, v_n = 0 \text{ for n sufficiently negative}\} \tag{1.5}$$

$V[[x,x^{-1}]]$ is the space of *formal Laurent series*, $V[x,x^{-1}]$ is the space of *formal Laurent polynomials*, $V[[x]]$ is the space of *formal power series*, and $V((x))$ is the space of *truncated formal Laurent series.*

1.2 Formal limits

First we will need to define the idea of *summability* of series of endomorphisms of a vector space V. As usual, the set of endomorphisms of a vector space V is denoted by End V.

Definition 1.1. Let V be a vector space and let $(f_i)_{i\in I}$ be a family in End V. The family $(f_i)_{i\in I}$ is *summable* if for all $v \in V$, $f_i(v) = 0$ for all but finitely many $i \in I$.

Note that this notion of summability is more restrictive than the idea of summability in $\mathbb{R}^n$ from analysis. We don't allow for convergent series where infinitely many terms are nonzero, but instead only consider sums that when applied to any specific vector $v \in V$ are finite (but not every endomorphism necessarily vanishes). Consider the following examples.

Example 1.2. Let the vector space be $V = \mathbb{C}$ and let the indexing set be $I = \mathbb{C}$. Define the family $(f_w)_{w\in\mathbb{C}}$ by

$$f_w(z) \equiv \delta_{w,z}, \tag{1.6}$$

where $\delta_{w,z}$ is the Kronecker delta function. (Recall that the Kronecker delta function is defined by $\delta_{w,z} = 1$ if $w = z$ and $\delta_{w,z} = 0$ if $w \neq z$.) Then for each $z \in \mathbb{C}$, precisely one of the $f_w(z)$ is non-vanishing: namely, $f_z(z) = 1$. Thus, the family is summable and equals

$$\sum_{w\in\mathbb{C}} f_w(z) = f_z(z) = z \tag{1.7}$$

for every $z \in \mathbb{C}$, i.e. the sum is the identity endomorphism $\mathrm{id}_{\mathbb{C}} \in \operatorname{End} \mathbb{C}$. (For future reference the identity endomorphism of a vector space V will be denoted by $\mathrm{id}_V \in \operatorname{End} V$.)

Nonexample 1.3. Let $V = \mathbb{C}$ and let our indexing set be $I = \mathbb{Z}$. Define

$$f_n = \begin{cases} 1 & : \text{ if } n = 0 \\ \dfrac{1}{n^2} & : \text{ otherwise.} \end{cases} \tag{1.8}$$

Let $n \in \mathbb{Z}$ be an arbitrary fixed integer. We have

$$\sum_{n\in\mathbb{Z}} f_n = 1 + \sum_{n\neq 0} \frac{1}{n^2} = 1 + \sum_{n<0} \frac{1}{n^2} + \sum_{n>0} \frac{1}{n^2} = 1 + 2\sum_{n>0} \frac{1}{n^2} \tag{1.9}$$

$$= 1 + \frac{\pi^2}{3} \tag{1.10}$$

if we apply the usual notion of summability in the case of complex numbers. However, this family is not summable under our new definition, since e.g. $z = 1 \in \mathbb{Z}$, all the $f_n(1)$ are nonzero.

Nonexample 1.3 illustrates how our new notion of summability differs from that used in analysis for $\mathbb{R}^n$. Even though the series converges in the usual sense, it is not considered summable. While this more restrictive definition limits our examples in the case where the base field is $k = \mathbb{C}$, it allows us to easily generalize to other base fields k in the definition of a vertex operator algebra. Next we will define another notion of summability that applies to formal power series in $(\text{End } V)[[x, x^{-1}]]$ rather than just series in End V.

Definition 1.4. Let $(F_i(x))_{i \in I}$ be a family in $(\text{End } V)[[x, x^{-1}]]$ where for each $i \in I$, set

$$F_i(x) = \sum_{n \in \mathbb{Z}} f_i(n) x^n. \tag{1.11}$$

Then the family $(F_i(x))_{i \in I}$ is *summable* if for all $n \in \mathbb{Z}$, the family of coefficients $(f_i(n))_{i \in I}$ is summable. The sum equals

$$\sum_{i \in I} F_i(x) = \sum_{n \in \mathbb{Z}} \left(\sum_{i \in I} f_i(n) \right) x^n. \tag{1.12}$$

Remark 1.5. In the case $|I| = 1$ ($|I|$ here denotes the cardinality of the set I), we have a single formal series

$$\sum_{n \in \mathbb{Z}} f(n) x^n \in (\text{End } V)[[x, x^{-1}]], \tag{1.13}$$

and the family $(f(n)x^n)_{n \in \mathbb{Z}}$ is summable; its sum equals $F(x)$ because there is only one term in the family for each power of x. This generalizes to the case where the cardinality of I is finite, as there are only finitely many nonzero endomorphisms $f_i(n)$ for a given power in x.

In addition to sums, we also want to know when a product of series exists.

Definition 1.6. Let $(F_i(x))_{i=1}^{r}$ be a finite family in $(\text{End } V)[[x, x^{-1}]]$, where as before we set

$$F_i(x) = \sum_{n \in \mathbb{Z}} f_i(n) x^n. \tag{1.14}$$

Then we say the product $F_1(x)\cdots F_r(x)$ *exists* if for all $n \in \mathbb{Z}$, the family

$$(f(n_1)\cdots f(n_r))_{n_1,\ldots,n_r;n_1+\cdots+n_r=n} \tag{1.15}$$

is summable. The product evaluates to

$$F_1(x)\cdots F_r(x) = \sum_{n\in\mathbb{Z}}\left(\sum_{n_1+\cdots+n_r=n} f_1(n_1)\cdots f_r(n_r)\right)x^n \in (\mathrm{End}\ V)[[x,x^{-1}]] \tag{1.16}$$

when it exists.

Consider the following example of a product.

Example 1.7. Set $r = 2$. Let $V = \mathbb{C}$. Define $F_1(x)$ and $F_2(x)$ by

$$(F_1(x))(z) \equiv \sum_{n\in\mathbb{Z}} \delta_{n,z}x^n \in \mathbb{C}[[x,x^{-1}]] \tag{1.17}$$

and

$$(F_2(x))(z) \equiv \sum_{n\in\mathbb{Z}} \delta_{n+1,z}x^n \in \mathbb{C}[[x,x^{-1}]] \tag{1.18}$$

for each $z \in \mathbb{C}$. Then we have two corresponding families of coefficients $(f_1(n_1))_{n_1\in\mathbb{Z}}$ and $(f_2(n_2))_{n_2\in\mathbb{Z}}$ defined by $(f_1(n_1))(z) \equiv \delta_{n_1,z}$ and $(f_2(n_2))(z) \equiv \delta_{n_2+1,z}$. We need to show that the family $(f_1(n_1)f_2(n_2))_{n_1,n_2;n_1+n_2=n}$ is summable for each $n \in \mathbb{Z}$. We have

$$\sum_{n_1+n_2=n}(f_1(n_1)f_2(n_2))(z) = \sum_{n_1+n_2=n}(f_1(n_1))(z)(f_2(n_2))(z) = \sum_{n_1+n_2=n}\delta_{n_1,z}\delta_{n_2+1,z} \tag{1.19}$$

$$= \begin{cases} 1 & : \text{ if } n \text{ odd and } z = \dfrac{n+1}{2} \\ 0 & : \text{ otherwise.} \end{cases} \tag{1.20}$$

This is since $\delta_{n_1,z} = 1$ only if $n_1 = z$ and $\delta_{n_2+1,z} = 1$ only if $n_2 + 1 = z$. Thus $n_1 = n_2 + 1$

for a nonvanishing term, which implies $n = 2n_2 + 1$, so n must be odd for the sum to not vanish. Moreover, substituting in the equation $z = n_1$ gives us the additional requirement $z = (n+1)/2$. Since the coefficient is nonvanishing only for one choice of n_1 and n_2, this family is clearly summable. Therefore, the product of formal series is summable and becomes

$$(F_1(x)F_2(x))(z) = \left(\sum_{n\in\mathbb{Z}}\left(\sum_{n_1+n_2=n} f_1(n_1)f_2(n_2)\right)x^n\right)(z) \tag{1.21}$$

$$= \sum_{n\in\mathbb{Z}}\left(\sum_{n_1+n_2=n} (f_1(n_1)f_2(n_2))(z)\right)x^n \tag{1.22}$$

$$= \sum_{n \text{ even}}\left(\sum_{n_1+n_2=n} (f_1(n_1)f_2(n_2))(z)\right)x^n \tag{1.23}$$

$$+ \sum_{n \text{ odd}}\left(\sum_{n_1+n_2=n} (f_1(n_1)f_2(n_2))(z)\right)x^n \tag{1.24}$$

$$= 0 + \sum_{n \text{ odd}}\left(\sum_{n_1+n_2=n} (f_1(n_1)f_2(n_2))(z)\right)x^n \tag{1.25}$$

$$= \begin{cases} 1 & : \text{ if } z \in \mathbb{Z} \\ 0 & : \text{ otherwise.} \end{cases} \tag{1.26}$$

Now we will define the notion of a *formal limit*, which is used in the definition of a vertex algebra. This is similar to the usual notion of limit from analysis, although here we simply interpret a limit as a coordinate change that satisfies a summability condition to ensure the resulting series exists. As a result, we can make sense of this definition outside of the case $k = \mathbb{C}$ or $k = \mathbb{R}$.

Definition 1.8 (Formal limit)**.** Let

$$\sum_{m,n\in\mathbb{Z}} F(m,n)x_1^m x_2^n \in (\text{End } V)[[x_1, x_1^{-1}, x_2, x_2^{-1}]]. \tag{1.27}$$

The *formal limit*

$$\lim_{x_1 \to x_2} \sum_{m,n \in \mathbb{Z}} F(m,n) x_1^m x_2^n \tag{1.28}$$

exists if for every $n \in \mathbb{Z}$, the family $(F(m, n-m))_{m \in \mathbb{Z}}$ is summable. If it exists, the limit equals

$$\lim_{x_1 \to x_2} \sum_{m,n \in \mathbb{Z}} F(m,n) x_1^m x_2^n = \sum_{n \in \mathbb{Z}} \left(\sum_{m \in \mathbb{Z}} F(m, n-m) \right) x_2^n. \tag{1.29}$$

We have the following simple example of a formal limit using this purely algebraic definition.

Example 1.9. Let $V = \mathbb{C}$. Define $F(m,n) \equiv \delta_{m,n}$ for all $m, n \in \mathbb{Z}$. We will show that the limit

$$\lim_{x_1 \to x_2} \sum_{m,n \in \mathbb{Z}} F(m,n) x_1^m x_2^n \tag{1.30}$$

exists. To do so we need to show that the family $(F(m, n-m))_{m \in \mathbb{Z}}$ is summable. We have

$$(F(m, n-m))(z) = \delta_{m,n-m} z = \begin{cases} z & : \text{ if } 2m = n \\ 0 & : \text{ otherwise.} \end{cases} \tag{1.31}$$

Note that this is nonzero for only one value of m if n is even and none if n is odd, therefore the family of coefficients is summable and we conclude the formal limit (1.30) exists. We then evaluate the formal limit as

$$\left(\lim_{x_1\to x_2}\sum_{m,n\in\mathbb{Z}}F(m,n)x_1^m x_2^n\right)(z)=\left(\sum_{n\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}}F(m,n-m)\right)x_2^n\right)(z) \tag{1.32}$$

$$=\sum_{n\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}}(F(m,n-m))(z)\right)x_2^n \tag{1.33}$$

$$=\sum_{n\text{ odd}}\left(\sum_{m\in\mathbb{Z}}(F(m,n-m))(z)\right)x_2^n \tag{1.34}$$

$$+\sum_{n\text{ even}}\left(\sum_{m\in\mathbb{Z}}(F(m,n-m))(z)\right)x_2^n \tag{1.35}$$

$$=\sum_{n\text{ even}} zx_2^n. \tag{1.36}$$

1.3 The formal residue operator

Now we introduce the formal residue operator. In the case $k=\mathbb{C}$ this is precisely the usual notion of residue in complex analysis calculated from a Laurent series expansion. Here the definition is extended in a formal way to Laurent series with coefficients in an arbitrary field k.

Definition 1.10 (Formal residue operator)**.** The *formal residue operator*, denoted $\mathrm{Res}_x : V[[x,x^{-1}]]\to V$, is defined by

$$\mathrm{Res}_x v(x)\equiv \text{ the coefficient of } x^{-1} \text{ in } v(x) \tag{1.37}$$

for a series $v(x)\in V[[x,x^{-1}]]$.

The following is an example of the above definition.

Example 1.11. Let $V = \mathbb{C}$. Define

$$v(x) \equiv \sum_{n\in\mathbb{Z}} x^n \in \mathbb{C}[[x, x^{-1}]]. \tag{1.38}$$

Then we apply the formal residue operator $\mathrm{Res}_x : \mathbb{C}[[x, x^{-1}]] \to \mathbb{C}$ to $v(x)$ and get

$$\mathrm{Res}_x v(x) = \mathrm{Res}_x \sum_{n\in\mathbb{Z}} x^n = 1. \tag{1.39}$$

As we will see in future sections, the formal residue operator has a simple definition yet is useful in proving the Jacobi identity holds for examples of vertex algebras.

1.4 The modified binomial expansion

We also introduce new notation for the modified binomial expansion to avoid confusion (as opposed to [LL04, 30-31]). It is challenging to work with the usual notation since the exponents are commonly used to mean something else in algebra, and, in particular, commutativity fails to hold when the exponent is negative ($B_n(x, y) \neq B_n(y, x)$ in the following Definition 1.12).

Definition 1.12 (Modified binomial expansion)**.** Define the map $B_n : \{(x, y)\} \to \mathbb{Q}[[x, x^{-1}, y, y^{-1}]]$ for each $n \in \mathbb{Z}$ by

$$B_n(x, y) \equiv \sum_{k\in\mathbb{N}} b_{nk} x^{n-k} y^k, \tag{1.40}$$

where we define the coefficients

$$b_{nk} \equiv \frac{n(n-1)\cdots(n-k+1)}{k!} \in \mathbb{Q}. \tag{1.41}$$

Remark 1.13. Note that in general $B_n(x, y) \neq B_n(y, x)$ due to the change in the coefficients from the usual definition of the binomial expansion. This is why we use a different notation from [LL04, 30-31]; the authors use $(x+y)^n$ to mean $B_n(x,y)$, and confusingly use the binomial coefficient symbols to mean the coefficients b_{nk}. Note, however, that the coefficients b_{nk} are the usual binomial coefficients in the case where n is nonnegative, and as a result $B_n(x,y) = B_n(y,x) = (x+y)^n$ where $(x+y)^n$ refers to the usual binomial expansion in the case $n \geq 0$.

Example 1.14. We compute $B_2(x,y)$ using the formula

$$B_2(x,y) = \sum_{k\in\mathbb{N}} b_{2k} x^{2-k} y^k = \sum_{k\in\mathbb{N}} \frac{2(1)\cdots(2-k+1)}{k!} x^{2-k} y^k \tag{1.42}$$

$$= \frac{2!}{0!(2-0)!} x^{2-0} y^0 + \frac{2!}{1!(2-1)!} x^{2-1} y^1 + \frac{2!}{2!(2-2)!} = x^2 + 2xy + y^2 \tag{1.43}$$

where we used the fact that

$$b_{nk} = \binom{n}{k} \tag{1.44}$$

for $n \geq 0$. (Note that if $k = 0$ we simply set the coefficient to 1 and get the term x^2.) Reversing the roles of x and y in $(1.42-43)$ shows that $B_2(x,y) = B_2(y,x)$.

Example 1.15. We can also consider the case where n is negative, say $n = -2$. Then $B_{-2}(x,y)$ equals

$$B_{-2}(x,y) = \sum_{k\in\mathbb{N}} b_{-2,k} x^{-2-k} y^k = \sum_{k\in\mathbb{N}} \frac{(-2)(-3)\cdots(-2-k+1)}{k!} x^{-2-k} y^k \tag{1.45}$$

$$= x^{-2-0} y^0 + \frac{-2}{1!} x^{-2-1} y^1 + \frac{(-2)(-3)}{2!} x^{-2-2} y^2 + \cdots \tag{1.46}$$

$$= x^{-2} - 2x^{-3} y + 3x^{-4} y^2 + \cdots . \tag{1.47}$$

Example 1.16. Now we will show as an example that $B_{-1}(x,y) \neq B_{-1}(y,x)$. We have

$$B_{-1}(x,y) = \sum_{k\in\mathbb{N}} b_{-1,k} x^{-1-k} y^k = \sum_{k\in\mathbb{N}} \frac{(-1)\cdots(-k)}{k!} x^{-1-k} y^k \tag{1.48}$$

$$= \sum_{k\in\mathbb{N}} (-1)^k x^{-1-k} y^k. \tag{1.49}$$

Reversing the roles of x and y results in

$$B_{-1}(y,x) = \sum_{k\in\mathbb{N}} (-1)^k x^k y^{-1-k}, \tag{1.50}$$

so clearly $B_{-1}(x,y) \neq B_{-1}(y,x)$.

1.5 The formal delta function

Now we define the *formal delta function.*

Definition 1.17. The *formal delta function* is a function $\delta : \{x\} \to \mathbb{C}[[x,x^{-1}]]$

$$\delta(x) \equiv \sum_{n\in\mathbb{Z}} x^n \in \mathbb{C}[[x,x^{-1}]]. \tag{1.51}$$

Remark 1.18. We can extend the argument to expressions of the form e.g. $x-y$ by using the modified binomial expansion convention. So

$$\delta(x-y) = \sum_{n\in\mathbb{Z}} B_n(x,-y). \tag{1.52}$$

The formal delta function is necessary to define the Jacobi identity, which is the most important (and complicated) axiom of a vertex algebra. In future sections we will prove various useful identities involving the formal delta function.

We have the following basic result involving the formal delta function.

Theorem 1.19. (Proposition 2.1.8 of [LL04])

a Let $f(x) \in V[x, x^{-1}]$. Then $f(x)\delta(x) = f(1)\delta(x)$.

b Let

$$f(x_1, x_2) \in (\text{End } V)[[x_1, x_1^{-1}, x_2, x_2^{-1}]] \tag{1.53}$$

be so that

$$\lim_{x_1 \to x_2} f(x_1, x_2) \tag{1.54}$$

exists. Then we have

$$f(x_1, x_2)\delta\left(\frac{x_1}{x_2}\right) = f(x_1, x_1)\delta\left(\frac{x_1}{x_2}\right) = f(x_2, x_2)\delta\left(\frac{x_1}{x_2}\right) \tag{1.55}$$

and the expressions exist.

Proof. We get (a) by linearity and the fact that $x^n\delta(x) = \delta(x)$ for all $n \in \mathbb{Z}$. Now we will show that (b) holds. Let

$$f(x_1, x_2) = \sum_{m,n \in \mathbb{Z}} a(m, n) x_1^m x_2^n. \tag{1.56}$$

We then have

$$f(x_1, x_2)\delta\left(\frac{x_1}{x_2}\right) = \left(\sum_{m,n \in \mathbb{Z}} a(m, n) x_1^m x_2^n\right)\left(\sum_{k \in \mathbb{Z}} x_1^k x_2^{-k}\right) \tag{1.57}$$

$$= \sum_{m,n,k \in \mathbb{Z}} a(m, n) x_1^{m+k} x_2^{n-k} \tag{1.58}$$

$$= \sum_{m,n,k \in \mathbb{Z}} a(m, n) x_2^{m+n} x_1^{m+k} x_2^{-m-k} \tag{1.59}$$

$$= f(x_2, x_2)\delta\left(\frac{x_1}{x_2}\right), \tag{1.60}$$

and it is clear that the expressions exist. By $\delta(x_1/x_2) = \delta(x_2/x_1)$ the other equality follows. $\square$

1.6 The exponential map

Now we will introduce formal exponentials, which are defined by the familiar series expression for the exponential function, generalized to an arbitrary base field k. For the purposes of working with vertex algebras, we don't care about convergence where the formal variable gets replaced by a field element (e.g., a complex number). We only care that the formal series exists and proceed using only algebra whenever possible.

Definition 1.20. Let $S \in y(\mathrm{End}\ V)[[y]]$, i.e. S has no constant term. Then define the *exponential* of S by the formal series

$$e^S = \sum_{n \in \mathbb{N}} \frac{1}{n!} S^n. \tag{1.61}$$

Remark 1.21. As with the formal delta function, we extend the above definition to expressions where S is a difference of terms using our modified binomial expansion convention, replacing S^n with the appropriate expression of the form $B_n(x, y)$ for each term.

Like the usual exponential map, the formal exponential satisfies $e^{S+T} = e^S e^T$ where S and T are commuting elements of $y(\mathrm{End}\ V)[[y]]$.

Theorem 1.22. Let $S, T \in y(\mathrm{End}\ V)[[y]]$. Then $e^{S+T} = e^S e^T$.

Proof. The left hand side becomes

$$e^{S+T} = \sum_{n \in \mathbb{N}} \frac{1}{n!} B_n(S, T) = \sum_{n \in \mathbb{N}} \left(\frac{1}{n!} \sum_{k \in \mathbb{N}} b_{nk} S^{n-k} T^k \right). \tag{1.62}$$

On the right hand side we have

$$e^S e^T = \left(\sum_{n\in\mathbb{N}} \frac{1}{n!}S^n\right)\left(\sum_{n\in\mathbb{N}} \frac{1}{n!}T^n\right) = \sum_{n\in\mathbb{N}} \sum_{n_1+n_2=n} \frac{1}{n_1!n_2!} S^{n_1}T^{n_2} \tag{1.63}$$

$$= \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \frac{1}{(n-k)!k!} S^{n-k}T^k = \sum_{n\in\mathbb{N}} \left(\frac{1}{n!}\sum_{k\in\mathbb{N}} b_{nk} S^{n-k}T^k\right). \tag{1.64}$$

The result then follows immediately. □

We will need the definition of a derivation to prove the next theorem. Intuitively, derivations are functions that act like derivatives.

Definition 1.23. A *derivation* on an algebra A over a field k is a linear map (over k) $d : A \to A$ that satisfies $d(ab) = ad(b) + d(a)b$ for all $a, b \in A$ (called Leibniz's law).

Theorem 1.24. Let

$$T \equiv p(x)\frac{d}{dx} \tag{1.65}$$

where $p(x) \in \mathbb{C}[x, x^{-1}]$. Then we have

$$e^{yT}(f(x)g(x)) = (e^{yT}f(x))(e^{yT}g(x)) \tag{1.66}$$

for all $f(x) \in \mathbb{C}[x, x^{-1}]$ and $g(x) \in \mathbb{C}[[x, x^{-1}]]$.

Proof. Since T is a derivation of $\mathbb{C}[x, x^{-1}]$ we have

$$T(f(x)g(x)) = (Tf(x))g(x) + f(x)(Tg(x)) \tag{1.67}$$

for all $f(x) \in \mathbb{C}[x, x^{-1}]$ and $g(x) \in \mathbb{C}[[x, x^{-1}]]$. By induction we obtain

$$T^n(f(x)g(x)) = \sum_{k=0}^{n} b_{nk}(T^k f(x))(T^{n-k}g(x)) \tag{1.68}$$

for $n \geq 0$, and so we have

$$e^{yT}(f(x)g(x)) = \sum_{n\in\mathbb{N}} \frac{1}{n!}(yT)^n(f(x)g(x)) = \sum_{n\in\mathbb{N}} \frac{1}{n!} y^n T^n(f(x)g(x)) \tag{1.69}$$

$$= \sum_{n\in\mathbb{N}} \frac{1}{n!} y^n \sum_{k=0}^{n} b_{nk}(T^k f(x))(T^{n-k} g(x)) \tag{1.70}$$

and

$$(e^{yT}f(x))(e^{yT}g(x)) = \left(\sum_{n\in\mathbb{N}} \frac{1}{n!}(yT)^n(f(x))\right)\left(\sum_{n\in\mathbb{N}} \frac{1}{n!}(yT)^n(g(x))\right) \tag{1.71}$$

$$= \sum_{n\in\mathbb{N}} \sum_{n_1+n_2=n} \frac{1}{n_1!n_2!}(yT)^{n_1}(f(x))(yT)^{n_2}(g(x)) \tag{1.72}$$

$$= \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} \frac{1}{k!(n-k)!} y^k (T^k f(x)) y^{n-k}(T^{n-k} g(x)) \tag{1.73}$$

$$= \sum_{n\in\mathbb{N}} \frac{1}{n!} y^n \sum_{k\in\mathbb{N}} b_{nk}(T^k f(x))(T^{n-k} g(x)), \tag{1.74}$$

showing they are equal. □

We have the following important result called the "formal Taylor theorem" (Proposition 2.2.2 of [LL04, 32]).

Theorem 1.25. (Proposition 2.2.2 of [LL04, 32]) Let $v(x) \in V[[x, x^{-1}]]$. Then

$$e^{yd/dx} v(x) = v(x+y). \tag{1.75}$$

Proof. Let

$$v(x) = \sum_{n\in\mathbb{Z}} v_n x^n \in V[[x, x^{-1}]]. \tag{1.76}$$

Then we have

$$e^{yd/dx}v(x) = \sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \frac{y^i}{i!}\left(\frac{d}{dx}\right)^i v_n x^n \tag{1.77}$$

$$= \sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \frac{y^i}{i!} n\cdots(n-i+1)v_n x^{n-i} \tag{1.78}$$

$$= \sum_{n\in\mathbb{Z}}\sum_{i\in\mathbb{N}} b_{n,i}v_n y^i x^{n-i} \tag{1.79}$$

$$= \sum_{n\in\mathbb{Z}} v_n B_n(x,y) = v(x+y). \tag{1.80}$$

□

Note that whenever we make a change of variables $x \mapsto x+y$ for $v(x) \in V[[x,x^{-1}]]$, we replace each x^n in the series with $B_n(x,y)$. As we will see in future sections, this convention applies to all the formal series in the definition of a vertex operator algebra, including the vertex operator map $Y(\cdot,x)$ when making the change of coordinates $Y(\cdot,x) \mapsto Y(\cdot,x+y)$ and the formal delta function when making the change of coordinates $\delta(x) \mapsto \delta(x+y)$. We were able to improve the notation for the binomial expansion convention itself, but in the case of $Y(\cdot,x)$ there isn't an easy fix to the notation.

1.7 Formal delta function identities

The goal of this section is to build up to proofs of the most useful elementary results for checking that a vertex operator algebra satisfies the axioms, especially the Jacobi identity. These include in particular the equations

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right) = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) \tag{1.81}$$

and

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right), \tag{1.82}$$

which we will use often in subsequent chapters. Recall that due to the modified binomial expansion convention being applied to expressions like $x_1 - x_2$, the left hand side of Equation (1.81) *does not* evaluate to zero.

The following two definitions formalize the idea of expansions of zero for a single variable and two variables, respectively.

Definition 1.26. Let $\iota_+ : \mathbb{C}(x) \hookrightarrow \mathbb{C}((x))$ be defined so that $\iota_+ f$ is the expansion of the rational function $f \in \mathbb{C}(x)$ as a formal Laurent series in x, and similarly let $\iota_- : \mathbb{C}(x) = \mathbb{C}(x^{-1}) \hookrightarrow \mathbb{C}((x^{-1}))$ be defined so that $\iota_- f$ is the expansion of f as a formal Laurent series in x^{-1}. Now define $\Theta : \mathbb{C}(x) \to \mathbb{C}[[x, x^{-1}]]$ by $f \mapsto \iota_+ f - \iota_- f$. We then call elements of Im Θ expansions of zero.

Definition 1.27. Set $S = \{x_1, x_2, x_1 \pm x_2\}$. Let $\mathbb{C}[x_1, x_2]_S$ be the subalgebra of the field of rational functions $\mathbb{C}(x_1, x_2)$ in the two variables x_1 and x_2 generated by $x_1^{\pm 1}$, $x_2^{\pm 1}$, and $(x_1 \pm x_2)^{-1}$. Define ι_{12} to be the linear map

$$\iota_{12} : \mathbb{C}[x_1, x_2]_S \to \mathbb{C}[[x_1, x_1^{-1}, x_2, x_2^{-1}]] \tag{1.83}$$

such that $\iota_{12}(f(x_1, x_2))$ is the formal Laurent series expansion of $f(x_1, x_2)$ involving only finitely many negative powers of x_2. Analogously, we define the linear map $\iota_{21}(f(x_1, x_2))$ using the opposite expansion. Define $\Theta(f) \equiv \iota_{12} f - \iota_{21} f$. Elements of Im Θ are called expansions of zero.

Note that our subalgebra $\mathbb{C}[x_1, x_2]_S$ is indeed a proper subset of $\mathbb{C}(x_1, x_2)$. For example, $1/(x_1 + 1) \in \mathbb{C}(x_1, x_2) - \mathbb{C}[x_1, x_2]_S$.

Lemma 1.28. The formal delta function is an expansion of zero. In particular,

$$\delta(x) = \Theta((1-x)^{-1}) = B_{-1}(1,-x) + B_{-1}(x,-1). \tag{1.84}$$

Proof. We have

$$\delta(x) = \sum_{n\in\mathbb{Z}} x^n = \left(\sum_{n\geq 0} x^n\right) - \left(-\sum_{n<0} x^n\right) = \iota_+((1-x)^{-1}) - \iota_-((1-x)^{-1}) \tag{1.85}$$

$$= \Theta((1-x)^{-1}). \tag{1.86}$$

Also,

$$B_{-1}(1,-x) = \sum_{k\in\mathbb{N}} b_{-1,k} 1^{-1-k}(-x)^k = \sum_{k\in\mathbb{N}} \frac{-1(-1-1)\cdots(-1-k+1)}{k!}(-1)^k x^k \tag{1.87}$$

$$= \sum_{k\in\mathbb{N}} (-1)^k \frac{1(2)\cdots k}{k!}(-1)^k x^k = \sum_{k\in\mathbb{N}} x^k = \sum_{n\geq 0} x^n \tag{1.88}$$

and

$$B_{-1}(x,-1) = \sum_{k\in\mathbb{N}} b_{-1,k} x^{-1-k}(-1)^k = \sum_{k\in\mathbb{N}} \frac{-1(-1-1)\cdots(-1-k+1)}{k!} x^{-1-k}(-1)^k \tag{1.89}$$

$$= \sum_{k\in\mathbb{N}} (-1)^k \frac{1(2)\cdots k}{k!} x^{-1-k}(-1)^k = \sum_{k\in\mathbb{N}} x^{-1-k} = \sum_{n<0} x^n. \tag{1.90}$$

□

Lemma 1.29. We have

$$x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = \Theta((x_1-x_2)^{-1}) = B_{-1}(x_1,-x_2) + B_{-1}(x_2,-x_1). \tag{1.91}$$

Proof. Lemma 1.28 states that the formal delta function is an expansion of zero

$$\delta(x) = \Theta((1-x)^{-1}) = B_{-1}(1,-x) + B_{-1}(x,-1). \tag{1.92}$$

Then we have

$$x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = x_2^{-1}\Theta\left(\left(1-\frac{x_1}{x_2}\right)^{-1}\right) = x_2^{-1}\left[B_{-1}\left(1,-\frac{x_1}{x_2}\right) + B_{-1}\left(\frac{x_1}{x_2},-1\right)\right]. \tag{1.93}$$

We evaluate each term as

$$x_2^{-1}B_{-1}\left(1,-\frac{x_1}{x_2}\right) = x_2^{-1}\sum_{k\in\mathbb{N}} b_{-1,k}1^{-1-k}\left(\frac{-x_1}{x_2}\right)^k = x_2^{-1}\sum_{k\in\mathbb{N}}(-1)^k(-1)^k\left(\frac{x_1}{x_2}\right)^k \tag{1.94}$$

$$= \sum_{k\in\mathbb{N}} x_1^k x_2^{-1-k} = \sum_{k\in\mathbb{N}}(-1)^k x_2^{-1-k}(-x_1)^k = \sum_{k\in\mathbb{N}} b_{-1,k}x_2^{-1-k}(-x_1)^k \tag{1.95}$$

$$= B_{-1}(x_2,-x_1) \tag{1.96}$$

and

$$x_2^{-1}B_{-1}\left(\frac{x_1}{x_2},-1\right) = x_2^{-1}\sum_{k\in\mathbb{N}} b_{-1,k}\left(\frac{x_1}{x_2}\right)^{-1-k}(-1)^k = x_2^{-1}\sum_{k\in\mathbb{N}}(-1)^k\left(\frac{x_1}{x_2}\right)^{-1-k}(-1)^k \tag{1.97}$$

$$= \sum_{k\in\mathbb{N}} x_1^{-1-k}x_2^{-1+1+k} = \sum_{k\in\mathbb{N}} x_1^{-1-k}x_2^k = \sum_{k\in\mathbb{N}}(-1)^k x_1^{-1-k}(-x_2)^k \tag{1.98}$$

$$= \sum_{k\in\mathbb{N}} b_{-1,k}x_1^{-1-k}(-x_2)^k = B_{-1}(x_1,-x_2). \tag{1.99}$$

This proves the result. □

Lemma 1.30. (Proposition 2.3.6 of [LL04, 35]) For $n \in \mathbb{N}$,

$$\frac{1}{n!}\left(\frac{\partial}{\partial x_2}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = B_{-n-1}(x_1,-x_2) - B_{-n-1}(-x_2,x_1) = \frac{(-1)^n}{n!}\left(\frac{\partial}{\partial x_1}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right). \tag{1.100}$$

Proof. The proof is by induction. Note that for $n = 0$ the result follows immediately from Lemma 1.29. Now consider the case $n = 1$. Then

$$\frac{\partial}{\partial x_2} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = \frac{\partial}{\partial x_2}\Theta((x_1-x_2)^{-1}) = \Theta\left(\frac{\partial}{\partial x_2}(x_1-x_2)^{-1}\right) = \Theta((x_1-x_2)^{-2}), \tag{1.101}$$

$$-\frac{\partial}{\partial x_1} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = -\frac{\partial}{\partial x_1}\Theta((x_1-x_2)^{-1}) = -\Theta\left(\frac{\partial}{\partial x_1}(x_1-x_2)^{-1}\right) = -\Theta(-(x_1-x_2)^{-2}) \tag{1.102}$$

$$= \Theta((x_1-x_2)^{-2}). \tag{1.103}$$

Now assume the result holds for $n = k$. Then we have

$$\frac{1}{(k+1)!}\left(\frac{\partial}{\partial x_2}\right)^{k+1} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = \frac{1}{k+1}\frac{\partial}{\partial x_2}\Theta((x_1-x_2)^{-k-1}) = \frac{1}{k+1}\Theta\left(\frac{\partial}{\partial x_2}(x_1-x_2)^{-k-1}\right) \tag{1.104}$$

$$= \frac{1}{k+1}\Theta((k+1)(x_1-x_2)^{-k-2}) = \Theta((x_1-x_2)^{-k-2}) \tag{1.105}$$

and

$$\frac{(-1)^{k+1}}{(k+1)!}\left(\frac{\partial}{\partial x_1}\right)^{k+1} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = -\frac{1}{k+1}\frac{\partial}{\partial x_1}\Theta((x_1-x_2)^{-k-1}) = -\frac{1}{k+1}\Theta\left(\frac{\partial}{\partial x_1}(x_1-x_2)^{-k-1}\right) \tag{1.106}$$

$$= -\frac{1}{k+1}\Theta((-k-1)(x_1-x_2)^{-k-2}) = \Theta((x_1-x_2)^{-k-2}). \tag{1.107}$$

This completes the proof by induction. ∎

Now we are ready to prove our favorite result.

Theorem 1.31. (Proposition 2.3.8(ii) of [LL04, 37]) We have

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right). \tag{1.108}$$

Proof. We will use the fact from Lemma 1.30 that for $n \in \mathbb{N}$,

$$\frac{1}{n!}\left(\frac{\partial}{\partial x_2}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = B_{-n-1}(x_1, -x_2) - B_{-n-1}(-x_2, x_1) = \frac{(-1)^n}{n!}\left(\frac{\partial}{\partial x_1}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right). \tag{1.109}$$

If we multiply by a factor of x_0^n and take the sum over $n \geq 0$, the middle expression becomes

$$\sum_{n\geq 0} x_0^n(B_{-n-1}(x_1, -x_2) - B_{-n-1}(-x_2, x_1)) = \sum_{n\in\mathbb{Z}} x_0^n(B_{-n-1}(x_1, -x_2) - B_{-n-1}(-x_2, x_1)) \tag{1.110}$$

$$= x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right), \tag{1.111}$$

where the last equality follows by definition of the formal delta function. The expression on the right becomes

$$\sum_{n\in\mathbb{N}} \frac{x_0^n}{n!}\left(-\frac{\partial}{\partial x_1}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = e^{-x_0\partial/\partial x_1} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right). \tag{1.112}$$

Define $v(x_1) \equiv x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) \in \mathbb{C}[[x_1, x_2, x_1^{-1}, x_2^{-1}]]$. By Theorem 1.22 we have

$$e^{-x_0\partial/\partial x_1} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = e^{-x_0\partial/\partial x_1} v(x_1) = v(x_1 + -x_0) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right), \tag{1.113}$$

proving the result. □

Now we will show that the other important identity involving the formal delta function holds.

Theorem 1.32. (Proposition 2.3.8(i) of [LL04, 37]) We have

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right). \tag{1.114}$$

Proof. Recall from Lemma 1.30 that we have shown

$$\frac{1}{n!}\left(\frac{\partial}{\partial x_2}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = B_{-n-1}(x_1, -x_2) - B_{-n-1}(-x_2, x_1) = \frac{(-1)^n}{n!}\left(\frac{\partial}{\partial x_1}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) \tag{1.115}$$

for all $n \in \mathbb{N}$. Summing the left-hand side multiplied by x_0^n over $n \in \mathbb{N}$ gives

$$\sum_{n\in\mathbb{N}} \frac{x_0^n}{n!}\left(\frac{\partial}{\partial x_2}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = e^{x_0\partial/\partial x_2} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = e^{x_0\partial/\partial x_2} x_1^{-1}\delta\left(\frac{x_2}{x_1}\right) \tag{1.116}$$

$$= x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right). \tag{1.117}$$

Similarly, on the right-hand side we have

$$\sum_{n\in\mathbb{N}} \frac{(-1)^n x_0^n}{n!} \left(\frac{\partial}{\partial x_1}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = \sum_{n\in\mathbb{N}} \frac{x_0^n}{n!} \left(-\frac{\partial}{\partial x_1}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) \tag{1.118}$$

$$= e^{-x_0\partial/\partial x_1} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right). \tag{1.119}$$

This proves the result. □

Now we will show two more useful results involving the formal delta function and formal Laurent polynomials and series.

Lemma 1.33. (Proposition 2.3.21 of [LL04]) Let

$$p(x_1, x_2) \in \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}] \tag{1.120}$$

and

$$f(x_1, x_2) \in (\text{End } V)[[x_1, x_1^{-1}, x_2, x_2^{-1}]] \tag{1.121}$$

so that the formal limit

$$\lim_{x_1\to x_2} f(x_1, x_2) \tag{1.122}$$

exists. Let

$$T = p(x_1, x_2)\frac{\partial}{\partial x_1}. \tag{1.123}$$

Then we have

$$f(x_1, x_2)e^{yT}\delta\left(\frac{x_1}{x_2}\right) = (e^{-yT}f)(x_1, x_2)e^{yT}\delta\left(\frac{x_1}{x_2}\right). \tag{1.124}$$

For $n \geq 0$ this is equivalent to the equation

$$f(x_1, x_2)T^n\delta\left(\frac{x_1}{x_2}\right) = \sum_{k=0}^{n}(-1)^k b_{n,k}(T^k f)(x_2, x_2)T^{n-k}\delta\left(\frac{x_1}{x_2}\right), \tag{1.125}$$

where all the expressions exist (either in the sense of a formal limit or a product of series where applicable).

Proof. Since

$$\lim_{x_1 \to x_2} \left(\frac{\partial f}{\partial x_1} \right) (x_1, x_2) \tag{1.126}$$

exists, by induction

$$\lim_{x_1 \to x_2} \left(\left(\frac{\partial}{\partial x_1} \right)^k f \right) (x_1, x_2) \tag{1.127}$$

exists for $k \geq 0$. This implies that

$$\lim_{x_1 \to x_2} (T^k f)(x_1, x_2) \tag{1.128}$$

exists for $k \geq 0$ since T^k is a polynomial in $\partial/\partial x_1$ with coefficients in $\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]$. Thus the expression on the right hand side of (1.125) exists in the sense of a product of formal series, showing that the expression on the right hand side of (1.124) exists. Similarly, we see that the left hand side of (1.125) exists, and so the left hand side of (1.124) exists. By Theorems 1.19 and 1.22, $e^S e^{-S} = 1$ for $S \in y(\text{End } V)[[y]]$. We then have $e^{yT}(f(x_1, x_2)g(x_1, x_2)) = (e^{yT} f(x_1, x_2))(e^{yT} g(x_1, x_2))$ (Theorem 1.23), and obtain

$$f(x_1, x_2) e^{yT} \delta \left(\frac{x_1}{x_2} \right) = e^{yT} \left[(e^{-yT} f)(x_1, x_2) \delta \left(\frac{x_1}{x_2} \right) \right] \tag{1.129}$$

$$= e^{yT} \left[(e^{-yT} f)(x_2, x_2) \delta \left(\frac{x_1}{x_2} \right) \right] \tag{1.130}$$

$$= (e^{-yT} f)(x_2, x_2) e^{yT} \delta \left(\frac{x_1}{x_2} \right). \tag{1.131}$$

□

We have the following generalization of the previous result.

Corollary 1.34. (Remark 2.3.24 of [LL04]) Let

$$f(x_1, x_2, y) \in (\mathrm{End}\, V)[[x_1, x_1^{-1}, x_2, x_2^{-1}]] \tag{1.132}$$

such that

$$\lim_{x_1 \to x_2} f(x_1, x_2, y) \tag{1.133}$$

exists and for all $v \in V$ we have

$$f(x_1, x_2, y)v \in V[[x_1, x_1^{-1}, x_2, x_2^{-1}]]((y)). \tag{1.134}$$

Then the result of Lemma 1.32 still applies:

$$f(x_1, x_2, y)e^{yT}\delta\left(\frac{x_1}{x_2}\right) = (e^{-yT}f)(x_2, x_2, y)e^{yT}\delta\left(\frac{x_1}{x_2}\right). \tag{1.135}$$

Proof. Apply the proof of Lemma 1.32 to $f(x_1, x_2, y_1)$, and then take $\lim_{y_1 \to y}$. □

Corollary 1.35. (Remark 2.3.25 of [LL04]) Let $f(x_1, x_2, y)$ meet the conditions of Corollary 1.33. Then

$$x_2^{-1}\delta\left(\frac{x_1 + y}{x_2}\right) f(x_1, x_2, y) = x_2^{-1}\delta\left(\frac{x_1 + y}{x_2}\right) f(x_2 - y, x_2, y), \tag{1.136}$$

$$x_1^{-1}\delta\left(\frac{x_2 - y}{x_1}\right) f(x_1, x_2, y) = x_1^{-1}\delta\left(\frac{x_2 - y}{x_1}\right) f(x_2 - y, x_2, y). \tag{1.137}$$

Proof. (1.115) follows from Corollary 1.33 where we set $p(x_1, x_2) = 1$ and the formal Taylor theorem once we multiply by x_2^{-1}. Then by Theorem 1.28 we obtain (1.116). □

Theorem 1.36. (Proposition 2.3.26 of [LL04]) Consider a formal Laurent series of the form

$$f(x_0, x_1, x_2) = \frac{g(x_0, x_1, x_2)}{x_0^r x_1^s x_2^t}, \tag{1.138}$$

where g is a polynomial and $r, s, t \in \mathbb{Z}$. Then

$$x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\iota_{20}(f|_{x_1=x_0+x_2}) = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\iota_{10}(f|_{x_2=x_1-x_0}) \tag{1.139}$$

and

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\iota_{12}(f|_{x_0=x_1-x_2}) - x_0^{-1}\delta\left(\frac{x_1-x_1}{-x_0}\right)\iota_{21}(f|_{x_0=x_1-x_2}) \tag{1.140}$$

$$= x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\iota_{10}(f|_{x_2=x_1-x_0}). \tag{1.141}$$

Proof. Multiply the formulas from Theorems 1.30 and 1.31 by $f(x_0, x_1, x_2)$ and note that by

$$x_1^{-1}\delta\left(\frac{x_2-y}{x_1}\right)f(x_1, x_2, y) = x_1^{-1}\left(\frac{x_2-y}{x_1}\right)f(x_2-y, x_2, y), \tag{1.142}$$

(Corollary 1.31) we have

$$x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)f(x_0, x_1, x_2) = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\iota_{20}(f|_{x_1=x_0+x_2}), \tag{1.143}$$

$$x_2^{-1}\delta\left(\frac{x_1+x_0}{x_2}\right)f(x_0, x_1, x_2) = x_2^{-1}\delta\left(\frac{x_1+x_0}{x_2}\right)\iota_{20}(f|_{x_2=x_1-x_0}), \tag{1.144}$$

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)f(x_0, x_1, x_2) = x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\iota_{20}(f|_{x_0=x_1-x_2}), \tag{1.145}$$

$$x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)f(x_0, x_1, x_2) = x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\iota_{20}(f|_{x_0=x_1-x_2}). \tag{1.146}$$

□

Remark 1.37. We can generalize the previous result in the following manner. Define the subalgebra A of the field of fractions of the ring $\mathbb{C}[[x_1, x_2]]$ by

$$A = \mathbb{C}((x_1, x_2))[(x_1 \pm x_2)^{-1}] = \mathbb{C}[[x_1, x_2]][x_1^{-1}, x_2^{-1}, (x_1 - x_2)^{-1}, (x_1 + x_2)^{-1}]. \tag{1.147}$$

We then extend the iota maps to the injections

$$\iota_{12} : A \hookrightarrow \mathbb{C}[[x_1, x_1^{-1}, x_2, x_2^{-1}]] \tag{1.148}$$

$$\iota_{21} : A \hookrightarrow \mathbb{C}[[x_1, x_1^{-1}, x_2, x_2^{-1}]] \tag{1.149}$$

and so on, which give the expansions involving only finitely many negative powers of x_2 and only finitely many negative powers of x_1, respectively. We then have the following generalization.

Theorem 1.38. (Proposition 2.3.27 of [LL04]) Let $f(x_0, x_1, x_2)$ be a formal Laurent series truncated from below in powers of x_0, x_1, and x_2, that is, $f(x_0, x_2, x_2) \in \mathbb{C}((x_0, x_1, x_2))$. Then the formulas of Theorem 1.35 still hold.

Proof. The proof is analogous to the proof of Theorem 1.35. □

Chapter 2: Finite dimensional vertex operator algebras

In this chapter I will classify finite dimensional vertex operator algebras as well as modules of finite dimensional vertex operator algebras. The example $V = \mathbb{C}$ will be used as a concrete special case. The results of this section apply to vertex operator algebras over an arbitrary field k since they depend only on linear algebra.

2.1 Definition of a vertex algebra

First we need to define the notion of a *vertex algebra*, to which we add a few additional axioms to reach the definition of a *vertex operator algebra*.

Definition 2.1 (Vertex algebra)**.** Let k be an arbitrary field. A *vertex algebra* (over k) is a vector space V together with a linear map

$$Y(\cdot, x) : V \to (\text{End } V)[[x, x^{-1}]] \tag{2.1}$$

$$v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \tag{2.2}$$

for each formal variable x where $v_n \in \text{End } V$ for all $n \in \mathbb{Z}$, and a chosen element $\mathbf{1} \in V$ called the *vacuum vector*. We call $Y(v, x)$ the *vertex operator associated with* v. The following axioms must be satisfied:

1. the *truncation condition* $Y(u, x)v \in V((x))$ for all $u, v \in V$,
2. the *vacuum property* $Y(\mathbf{1}, x) = \text{id}_V$,
3. the *creation property* $Y(v, x)\mathbf{1} \in V[[x]]$ and $\lim_{x \to 0} Y(v, x)\mathbf{1} = v$ for all $v \in V$, and

4. the *Jacobi identity*

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1) \tag{2.3}$$

$$=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2) \tag{2.4}$$

for all $u, v \in V$.

Remark 2.2. We will sometimes use the triple $(V, Y, \mathbf{1})$ to refer to a vertex algebra V with specified linear map Y and vacuum vector $\mathbf{1}$.

Definition 2.3. (Homomorphism of vertex algebras) Let $(V_1, Y_1, \mathbf{1}_1)$ and $(V_2, Y_2, \mathbf{1}_2)$ be vertex algebras. A *homomorphism* $f : V_1 \to V_2$ is a linear map such that

$$f(Y_1(u,x)v) = Y_2(f(u),x)f(v) \tag{2.5}$$

for $u, v \in V_1$ and $f(\mathbf{1}_1) = \mathbf{1}_2$. f is an *isomorphism* of vertex algebras if f is both a homomorphism and bijective. (Note that we are extending f canonically to $V_1[[x, x^{-1}]] \to V_2[[x, x^{-1}]]$ to make sense of the use of $Y_1(\cdot, x)$.)

2.2 Definition of a vertex operator algebra

Now we add a few axioms and data to the definition of a vertex algebra to get the vertex operator algebra definition, described below.

Definition 2.4 (Vertex operator algebra)**.** Let $(V, Y, \mathbf{1})$ be a vertex algebra over an arbitrary field k. This vertex algebra combined with a *conformal vector* $\omega \in V$ is a *vertex operator algebra* over k in the characteristic zero case if V has the $\mathbb{Z}$-grading

$$V = \bigoplus_{n\in\mathbb{Z}} V_{(n)} \tag{2.6}$$

with grading restrictions $\dim V_{(n)} < \infty$ for all $n \in \mathbb{Z}$ and $V_{(n)} = 0$ for n sufficiently negative. We define the notation wt $v \equiv n$ if $v \in V_{(n)}$. We require $\mathbf{1} \in V_{(0)}$ and $\omega \in V_{(2)}$. Moreover, the following additional axioms must be satisfied for $u, v \in V$:

1. the Virasoro algebra relations

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c_V \tag{2.7}$$

   for some $c_V \in \mathbb{C}$ called the *central charge* where we define $L(n)$ by

$$Y(\omega, x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}, \tag{2.8}$$

2. compatibility of $L(0)$ with the grading: $L(0)v = nv = (\text{wt } v)v$ for $n \in \mathbb{Z}$ and $v \in V_{(n)}$, and

3. the $L(-1)$-derivative property

$$Y(L(-1)v, x) = \frac{d}{dx}Y(v, x). \tag{2.9}$$

   In the case where k has prime characteristic, we replace the $\mathbb{Z}$-grading with a $\mathbb{Z}/p\mathbb{Z}$-grading where the copy $\mathbb{Z}/p\mathbb{Z} \subset k$ comes from adding or subtracting the multiplicative identity to itself in k:

$$V = \bigoplus_{n\in\mathbb{Z}/p\mathbb{Z}} V_{(n)}. \tag{2.10}$$

   Also, in the prime characteristic case we can get rid of the requirement $V_{(n)} = 0$ for n sufficiently negative, as it becomes redundant.

Remark 2.5. At first it may seem that vertex operator algebras are infinite dimensional in general due to the Virasoro algebra relations. However, this definition requires an *action* of the Virasoro rather than a copy as a subspace of V, so we can have the action of the Virasoro algebra on V send everything to zero, which turns out to characterize the finite dimensional case. Later in this chapter we will show that either $L(n) = 0$ for all $n \in \mathbb{Z}$ or $L(n) \neq 0$ for all $n \in \mathbb{Z}$, making any examples beyond the finite dimensional case very complicated.

Definition 2.6. Let V_1 and V_2 be vertex operator algebras. A homomorphism of vertex operator algebras $f : V_1 \to V_2$ is a homomorphism of V_1 to V_2 as vertex algebras satisfying $f(\omega_1) = \omega_2$, where ω_1 and ω_2 are the conformal vectors associated with V_1 and V_2, respectively. f is an isomorphism if it is both a homomorphism and bijective.

2.3 Classification of finite dimensional vertex operator algebras

Now we will consider the simplest family of examples of vertex operator algebras, namely the finite dimensional vertex operator algebras. In this case the vertex operator algebra structure can be determined by the axioms. First we show that the conformal vector $\omega = 0$ and the central charge $c_V = 0$ for any finite dimensional vertex operator algebra, then we derive the rest of the structure easily from it using a special case of Borcherds' construction of commutative vertex algebras.

Lemma 2.7 (Finite dimensional vertex operator algebras have $\omega = 0$ and $c_V = 0$)**.** Let $(V, Y, \mathbf{1}, \omega, c_V)$ be a finite dimensional vertex operator algebra. Then $\omega = 0$ and $c_V = 0$.

Proof. Let $V = \operatorname{span}\{e_i | 1 \leq i \leq d\}$. Recall from the definition of a vertex algebra that the truncation condition states that $Y(u, x)v \in V((x))$ for all $u, v \in V$. We set $u = \omega$ and $v = e_i$ and obtain $Y(\omega, x)e_i \in V((x))$. By the definition of $V((x))$ our series $Y(\omega, x)e_i$ is truncated, i.e. its terms vanish for sufficiently negative powers of x. Recall that we defined

$L(n)$ for $n \in \mathbb{Z}$ in the definition of a vertex operator algebra by

$$Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2}, \tag{2.11}$$

so this implies for all $1 \leq i \leq d$ there exists $n_i \in \mathbb{N}$ sufficiently large such that $L(n_i)e_i = 0$. Then we have

$$L(\max\{n_i | 1 \leq i \leq d\})e_j = 0 \tag{2.12}$$

for all $1 \leq j \leq d$, therefore $L(\max\{n_i | 1 \leq i \leq d\}) = 0$.

Now we show that if $L(n) = 0$ for some $n \in \mathbb{Z}$, then $L(m) = 0$ for all $m \in \mathbb{Z}$. We have just shown that $L(\max\{n_i | 1 \leq i \leq d\}) = 0$, so we set $n = \max\{n_i | 1 \leq i \leq d\}$ to satisfy the condition. Then we have $[L(m), L(n)] = [L(m), 0] = 0$, which implies

$$0 = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c_V \tag{2.13}$$

by the Virasoro algebra relations. If $m \neq -n$ then $\delta_{m+n,0} = 0$, so in this case the second term vanishes and we are left with $(m-n)L(m+n) = 0$. Then if $m = n$ we already know that $L(n) = 0$ from the first part of the proof, so the vanishing first term isn't a problem. Otherwise, we have $(m-n)L(m+n) = 0$ where $m - n \neq 0$, showing that $L(m+n) = 0$. Thus we have shown that $L(m+n) = 0$ if $m \neq -n$. If $m = -n$ then

$$[L(-n), L(n)] = -2nL(0) - \frac{1}{12}(n^3 - n)c_V = 0, \tag{2.14}$$

which implies

$$L(0) = -\frac{1}{12}\frac{n^3 - n}{2n}c_V. \tag{2.15}$$

This implies $L(0) = 0$ and $c_V = 0$ since n varies but $L(0)$ is fixed. Thus we have shown

$L(m+n)=0$ for all $m \in \mathbb{Z}$, or equivalently $L(m)=0$ for all $m \in \mathbb{Z}$.

The fact that all the $L(n)$ vanish implies that $Y(\omega, x)=0$. By the creation property we have

$$\lim_{x \to 0} Y(\omega, x)\mathbf{1} = \omega, \tag{2.16}$$

and since

$$\lim_{x \to 0} Y(\omega, x)\mathbf{1} = \lim_{x \to 0} 0\mathbf{1} = 0, \tag{2.17}$$

we conclude that $\omega = 0$. □

We can say more than this immediately about the vertex operator algebra structure in the finite dimensional case. The vertex operator $Y(\cdot, x)$ turns out to be constant.

Lemma 2.8. Let V be a finite dimensional vertex operator algebra. Then $Y(\cdot, x)$ is constant, i.e.

$$\frac{d}{dx} Y(v, x) = 0 \tag{2.18}$$

for all $v \in V$.

Proof. The result follows immediately from the fact that $L(-1)=0$ shown in the previous lemma and the $L(-1)$-derivative property:

$$\frac{d}{dx} Y(v, x) = Y(L(-1)v, x) = Y(0, x) = 0. \tag{2.19}$$

□

Lemma 2.9. Let V be a finite dimensional vertex operator algebra. Then $[Y(u, x_1), Y(v, x_2)] = 0$ for all $u, v \in V$.

Proof. We take the formal residue with respect to x_0 of both sides of the Jacobi identity, getting the left-hand side

$$\mathrm{Res}_{x_0}\left[x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2)-x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y(v,x_2)Y(u,x_1)\right] \tag{2.20}$$

$$=\mathrm{Res}_{x_0}\left[x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\right]Y(u,x_1)Y(v,x_2)-\mathrm{Res}_{x_0}\left[x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\right]Y(v,x_2)Y(u,x_1) \tag{2.21}$$

$$=Y(u,x_1)Y(v,x_2)-Y(v,x_2)Y(u,x_1)=[Y(u,x_1),Y(v,x_2)] \tag{2.22}$$

and the right-hand side

$$\mathrm{Res}_{x_0}\left[x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)\right]=\mathrm{Res}_{x_0}\left[x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\right]Y(Y(u,x_0)v,x_2) \tag{2.23}$$

$$=\left[\mathrm{Res}_{x_0}\left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\right)-\mathrm{Res}_{x_0}\left(x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\right)\right]Y(Y(u,x_0)v,x_2) \tag{2.24}$$

$$=(1-1)Y(Y(u,x_0)v,x_2)=0, \tag{2.25}$$

where we used Lemma 2.8 to pull Y out of the residue expressions since by Lemma 2.8 Y does not depend on the formal variable. Thus $[Y(u,x_1),Y(v,x_2)]=0$. □

Lemma 2.10 (Borcherds' construction)**.** Let V be a finite dimensional commutative associative algebra with identity $\mathbf{1}$. Then V is a vertex operator algebra with $\omega=0$, $c_V=0$, and $Y(u,x)v=uv$ for all $u,v\in V$.

Proof. Suppose V is a commutative associative algebra with identity $\mathbf{1}$ of dimension $0\leq d<\infty$. The forward direction of the proof amounts to a special case of Borcherds' construction of vertex operator algebras from commutative associative algebras with identity and a derivation, described in [Bor86]. We set $\omega=0$, $c_V=0$, $Y(u,x)v=uv$ for every $u,v\in V$ where the given product is that of V as a commutative associative algebra, $V_{(0)}=V$, and $V_{(n)}=0$ for $n\neq 0$. Now we show that with this structure V is made into a vertex operator algebra.

We have

$$Y(u,x)v = uv \in V \subset V((x)), \tag{2.26}$$

showing the truncation condition is satisfied. We also confirm that the vacuum property holds:

$$Y(\mathbf{1},x) = \mathrm{id}_V \tag{2.27}$$

by definition. Now we check the creation property. We have

$$Y(v,x)\mathbf{1} = v\mathbf{1} = v \in V \subset V[[x]] \tag{2.28}$$

and

$$\lim_{x\to 0} Y(v,x)\mathbf{1} = \lim_{x\to 0} v = v, \tag{2.29}$$

showing the creation property holds. The Jacobi identity holds since $Y(u,x_1)$ and $Y(v,x_2)$ commute (Lemma 2.9), and

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right) = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) \tag{2.30}$$

(Theorem 1.30). This shows that V is a vertex algebra.

Now we show that V is a vertex operator algebra. Note that $\dim V_{(0)} = 1 < \infty$ and $\dim V_{(n)} = 0 < \infty$ for $n \neq 0$. We also see that $V_{(n)} = 0$ for $n < 0$. We have $\mathbf{1} \in V = V_{(0)}$ and $\omega = 0 \in V_{(2)}$. We find that $L(n) = 0$ for all $n \in \mathbb{Z}$ since

$$0 = \omega = Y(\omega,x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}. \tag{2.31}$$

We then have

$$[L(m), L(n)] = 0 = 0 + 0 = L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c_V. \tag{2.32}$$

$L(0)$ is compatible with the grading since

$$L(0)v = 0v = 0 = (\text{wt } v)v \tag{2.33}$$

for all $v \in V$. Lastly, we check that the $L(-1)$-derivative property holds with

$$Y(L(-1)v, x) = 0 = \frac{d}{dx}v = \frac{d}{dx}Y(v, x). \tag{2.34}$$

Therefore V is a vertex operator algebra with the given structure. □

Now we use Borcherds' construction of a vertex algebra from a commutative associative algebra with derivation [Bor86] together with the remark [LL04, 84] that a vertex operator algebra with $\omega = 0$ is a commutative associative algebra to get the following classification of finite dimensional vertex operator algebras. We make use of one more lemma to prove associativity of the algebra structure (i.e. $Y(u \cdot v, x) = Y(u, x)Y(v, x)$), called the iterate formula.

Lemma 2.11 (The iterate formula)**.** We have

$$Y(Y(u, x_0)v, x_2) = \text{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1)\right). \tag{2.35}$$

Proof. This follows by applying Res_{x_1} to the Jacobi identity. The right hand side is obvious, and on the left hand side we have

$$\operatorname{Res}_{x_1}\left(x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v),x_2)\right)=\operatorname{Res}_{x_1}\left(x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y(Y(u,x_0)v,x_2)\right) \tag{2.36}$$

$$=Y(Y(u,x_0)v,x_2), \tag{2.37}$$

where we used Theorem 1.31 in the first equality. □

Theorem 2.12. *V is a finite dimensional commutative associative algebra over an arbitrary field k with identity $\mathbf{1}$ if and only if V is a finite dimensional vertex operator algebra over k. In particular, the conformal vector $\omega = 0$, the central charge $c_V = 0$, we have $V_{(0)} = V$ and $V_{(n)} = 0$ for $n \neq 0$, and*

$$Y(u,x)v \equiv uv \tag{2.38}$$

for all $u, v \in V$. Moreover, the isomorphism classes of finite dimensional vertex operator algebras over k correspond to the isomorphism classes of the commutative associative algebras over k with identity.

Proof. First, let V be a finite dimensional commutative associative algebra with identity $\mathbf{1}$. By Lemma 2.10 (Borcherds' construction) V is a vertex operator algebra with the stated structure. Conversely, suppose V is a finite dimensional vertex operator algebra with dimension $d < \infty$. Define the product

$$u \cdot v \equiv Y(u,x)v \tag{2.39}$$

for all $u, v \in V$. By the creation property and Lemma 2.8 we have

$$Y(v,x)\mathbf{1} = \lim_{x\to 0} Y(v,x)\mathbf{1} = v, \tag{2.40}$$

in other words $v \cdot \mathbf{1} = v$. We also get $\mathbf{1} \cdot v = 0$ directly from the vacuum property, therefore

$\mathbf{1}$ is an identity for $\cdot$. The associativity property for the scalar multiplication in $\mathbb{C}$ obviously holds since $Y(\cdot, x)$ is linear. Now we need to check that $\cdot$ is commutative. We have

$$u \cdot v = Y(u,x)v = Y(u,x)Y(v,x)\mathbf{1} = Y(v,x)Y(u,x)\mathbf{1} = Y(v,x)u = v \cdot u. \tag{2.41}$$

Associativity (i.e. $Y(u \cdot v, x) = Y(u,x)Y(v,x)$) follows from the iterate formula (Lemma 2.11). Therefore V is a commutative associative algebra with identity $\mathbf{1}$ under the product $\cdot$ and the usual addition for V as a vector space.

Now we check that an isomorphism of finite dimensional vertex operator algebras is also an isomorphism of their associated commutative associative algebras with identity. Recall that a vertex operator algebra homomorphism is a linear map $f : V_1 \to V_2$ such that

$$f(Y_1(u,x)v) = Y_2(f(u), x)f(v) \tag{2.42}$$

for all $u, v \in V_1$ [LL04, 98]. Using the definition of the product $u \cdot v = Y(u,x)v$, we can rewrite this condition as

$$f(u \cdot v) = f(u) \cdot f(v). \tag{2.43}$$

f is linear, so the conditions $f(u+v) = f(u) + f(v)$ and $f(\lambda u) = \lambda f(u)$ where $u, v \in V_1$ and $\lambda \in \mathbb{C}$ apply to both vertex operator algebra isomorphisms and associative algebra isomorphisms. Finally, if we let the vacuum vector $\mathbf{1}$ be the multiplicative identity, the condition $f(\mathbf{1}) = \mathbf{1}$ is the same for V_1 and V_2 as both vertex operator algebras and as commutative associative algebras with identity. Thus $f : V_1 \to V_2$ is also an isomorphism of V_1 and V_2 as commutative associative algebras with identity. The properties can clearly be reversed if we start with an isomorphism of algebras instead, showing the equivalence of the isomorphism classes. □

2.4 Example: $V = \mathbb{C}$

As a concrete example, we will now consider the case $V = \mathbb{C}$. First we will show that V can be given the structure of a vertex algebra, then that of a vertex operator algebra by satisfying additional axioms. It turns out that this is the simplest type of example, along with other finite dimensional cases, since it satisfies the Jacobi identity in the most trivial way, which becomes a simple identity of the formal delta function (Theorem 1.31).

Theorem 2.13. Define $Y(\cdot, x) : \mathbb{C} \to (\text{End } \mathbb{C})[[x, x^{-1}]]$ by sending $v \mapsto v = \sum_{n\in\mathbb{Z}} v_n x^{-n-1}$ where $v_n = v$ if $n = -1$ and $v_n = 0$ if $n \neq -1$. Then $(\mathbb{C}, Y, 1)$ has the structure of a vertex algebra.

Proof. We have

$$Y(u, x)v = uv \in \mathbb{C} \subset \mathbb{C}((x)), \tag{2.44}$$

showing the truncation condition is satisfied. We also confirm that the vacuum property holds:

$$Y(1, x) = 1 \tag{2.45}$$

by definition. Next we check the creation property. We have

$$Y(v, x)1 = v1 = v \in \mathbb{C} \subset \mathbb{C}[[x]] \tag{2.46}$$

and

$$\lim_{x\to 0} Y(v, x)1 = \lim_{x\to 0} v1 = v1 = v, \tag{2.47}$$

showing the creation property holds. Lastly, we show that the Jacobi identity holds. We have

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1) \tag{2.48}$$

$$=x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)uv-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)vu=uv\left[x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\right] \tag{2.49}$$

and

$$x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(uv,x_2)=uvx_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right). \tag{2.50}$$

Equality of (2.48) and (2.49) follows immediately from the equation

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right), \tag{2.51}$$

stated in Theorem 1.31. □

Now we will show $V=\mathbb{C}$ can also be given the structure of a vertex operator algebra.

Theorem 2.14. Consider the vertex algebra $(\mathbb{C},Y,1)$ from before and set $\omega\equiv 0\in\mathbb{C}$. Then $\mathbb{C}$ is a vertex operator algebra with central charge $c\equiv 0$.

Proof. We give $V=\mathbb{C}$ the grading

$$V=\bigoplus_{n\in\mathbb{Z}}V_{(n)} \tag{2.52}$$

where $V_{(0)}=V$ and $V_{(n)}=0$ for $n\neq 0$. Then dim $V_{(0)}=1<\infty$ and dim $V_{(n)}=0<\infty$ for $n\neq 0$. We also see that $V_{(n)}=0$ for $n<0$. We have $\mathbf{1}=1\in\mathbb{C}=V_{(0)}$ and $\omega=0\in V_{(2)}$.

We find that $L(n) = 0$ for all $n \in \mathbb{Z}$ since

$$0 = \omega = Y(\omega, x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}. \tag{2.53}$$

We then have

$$[L(m), L(n)] = 0 = 0 + 0 = L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c \tag{2.54}$$

where we set $c = 0$. $L(0)$ is compatible with the grading since

$$L(0)v = 0v = 0 = (\text{wt } v) \tag{2.55}$$

for all $v \in \mathbb{C}$. Lastly, we check that the $L(-1)$-derivative property holds with

$$Y(L(-1)v, x) = 0 = \frac{d}{dx}v = \frac{d}{dx}Y(v, x). \tag{2.56}$$

Therefore $\mathbb{C}$ is a vertex operator algebra with central charge $c = 0$. □

2.5 Definition of a vertex algebra module

Now we will define the notion of a *vertex algebra module*. As before, this definition applies to any choice of base field k.

Definition 2.15 (Vertex algebra module)**.** Let $(V, Y, \mathbf{1})$ be a vertex algebra over an arbitrary field k. A V*-module* is then a vector space W together with a linear map

$$Y_W(\cdot, x) : V \to (\text{End } W)[[x, x^{-1}]] \tag{2.57}$$

$$v \mapsto Y_W(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \tag{2.58}$$

for each formal variable x where $v_n \in \text{End } W$ for all $n \in \mathbb{Z}$. The following axioms must be satisfied: (1) the *truncation condition* $Y_W(u,x)w \in W((x))$, (2) the *vacuum property* $Y_W(\mathbf{1},x) = \text{id}_W$, and (3) the *Jacobi identity*

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u,x_1)Y_W(v,x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(v,x_2)Y_W(u,x_1) \tag{2.59}$$

$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y(u,x_0)v,x_2). \tag{2.60}$$

Now we note a general fact that every vertex algebra has a module, which we call the *adjoint module.*

Theorem 2.16 (Adjoint modules exist)**.** Let $(V, Y, \mathbf{1})$ be a vertex algebra. Define the linear map

$$Y_V(\cdot,x) : V \to (\text{End } V)[[x,x^{-1}]] \tag{2.61}$$

$$v \mapsto Y_V(v,x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \equiv Y(v,x) \tag{2.62}$$

Then (V, Y_V) is a V-module. (This is often called the *adjoint module.*)

Proof. The condition $u_n w = 0$ for n sufficiently large for $u, v \in V$ and $w \in W = V$ is just the truncation condition for V, which is satisfied by our assumption that V is a vertex algebra. The truncation condition and the vacuum property follow similarly by replacing Y_V in the expressions with Y. Therefore (V, Y_V) is a V-module. □

Definition 2.17. Let V be a vertex algebra. Let W_1 and W_2 be modules of the vertex algebra V (or V-modules where the term is unambiguous). A V-homomorphism $\psi : W_1 \to W_2$ is a linear map such that

$$\psi(Y_{W_1}(v,x)w) = Y_{W_2}(v,x)\psi(w) \tag{2.63}$$

for all $v \in V$ and $w \in W_1$. We denote the space of V-homomorphisms $W_1 \to W_2$ by $\mathrm{Hom}_V(W_1, W_2)$.

2.6 Definition of a vertex operator algebra module

Next we define the term *module of a vertex operator algebra*.

Definition 2.18 (Vertex operator algebra module)**.** Let V be a vertex operator algebra over an arbitrary field k. In the case where k is a characteristic zero field, a V-module is a module W for V viewed as a vertex algebra such that

$$W = \bigoplus_{h \in \mathbb{C}} W_{(h)}, \tag{2.64}$$

where $W_{(h)} = \{w \in W | L(0)w = hw\}$, the subspace of W of vectors of weight h, and such that the grading restriction conditions dim $W_{(h)} < \infty$ for $h \in \mathbb{C}$ and $W_{(h)} = 0$ for h whose real part is sufficiently negative. If k has prime characteristic, we replace the $\mathbb{C}$-grading with a $\mathbb{Z}/p\mathbb{Z}$-grading

$$W = \bigoplus_{h \in \mathbb{Z}/p\mathbb{Z}} W_{(h)}, \tag{2.65}$$

and remove the redundant requirement $W_{(h)} = 0$ for h whose real part is sufficiently negative.

Remark 2.19. In the case where k has prime characteristic, perhaps imposing a $\mathbb{Z}/p\mathbb{Z}$-grading is too restrictive compared to the $\mathbb{C}$-grading from the characteristic zero case. We wish to include adjoint modules as examples, so the grading should be over a field that contains our choice of field for grading (that is, $\mathbb{Z}/p\mathbb{Z}$) in the definition of a vertex operator algebra over a field of prime characteristic. The completion of the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$, denoted by $\widehat{\overline{\mathbb{Z}/p\mathbb{Z}}}$, may be a more suitable choice as the prime characteristic analogue to the $\mathbb{C}$-grading choice. Recall that the completion of a metric space is the result of adding points so that all Cauchy sequences converge to some point in the space.

Remark 2.20. Let V be a vertex operator algebra. Let W_1 and W_2 be V-modules, or modules of V as a vertex operator algebra. Then a V-homomorphism $\psi : W_1 \to W_2$ (as a homomorphism of modules of V as a vertex algebra) is compatible with the module grading in the sense that

$$\psi((W_1)_{(h)}) \subset (W_2)_{(h)} \tag{2.66}$$

for all $h \in \mathbb{C}$. This is because the condition $\psi(Y_{W_1}(v,x)w) = Y_{W_2}(v,x)\psi(w)$ is equivalent to $\psi(v_n w) = v_n \psi(w)$ for all $n \in \mathbb{Z}$, and so ψ commutes with $L(0)$. Thus, we get grading compatibility "for free" and do not need any additional requirements to define a homomorphism of modules of a vertex operator algebra compared to the more general vertex algebra case.

2.7 Classification of modules over finite dimensional vertex operator algebras

Now we classify the modules of a finite dimensional vertex operator algebra V.

Theorem 2.21. Let V be a finite dimensional vertex operator algebra. A vector space W is a module of the vertex operator algebra V if and only if W is finite dimensional with the $\mathbb{C}$-grading (or $\mathbb{Z}/p\mathbb{Z}$-grading in the prime characteristic case) defined by $W_{(0)} = W$ and $W_{(h)} = 0$ for $h \neq 0$, $Y_W(\cdot, x)$ is constant, $[Y_W(u,x_1), Y_W(v,x_2)] = 0$ for all $u, v \in V$, $Y_W(\mathbf{1}, x) = \mathrm{id}_W$, and $Y_W(u \cdot v, x) = Y_W(u,x)Y_W(v,x)$. Moreover, the isomorphism classes of such modules are equivalent to the isomorphism classes of modules over V as a commutative associative algebra with identity.

Proof. Let W be a module of the vertex operator algebra V. Then W has a $\mathbb{C}$-grading

$$W = \bigoplus_{h \in \mathbb{C}} W_{(h)} \tag{2.67}$$

or a $\mathbb{Z}/p\mathbb{Z}$-grading

$$W = \bigoplus_{h \in \mathbb{Z}/p\mathbb{Z}} W_{(h)} \tag{2.68}$$

in the case where k has prime characteristic where

$$W_{(h)} = \{w \in W | L(0)w = hw\} = \{w \in W | hw = 0\} \tag{2.69}$$

by the linearity of $Y_W(\cdot, x)$ and the fact that $L(0) = 0$. Thus we have $W_{(0)} = W$ and $W_{(h)} = 0$ for $h \neq 0$. By the grading restriction conditions we have dim $W =$ dim $W_{(0)} < \infty$. Using the result

$$Y_W(L(-1)v, x) = \frac{d}{dx} Y_W(v, x) \tag{2.70}$$

from [LL04] and the fact that $L(-1) = 0$, we have

$$\frac{d}{dx} Y_W(v, x) = 0 \tag{2.71}$$

for every $v \in V$. The last part of this direction of the proof is analogous to the trick involving the Jacobi identity in the proof of the previous theorem. We take the formal residue with respect to x_0 of both sides of the Jacobi identity, getting the left-hand side

$$\mathrm{Res}_{x_0} \left[x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u, x_1) Y_W(v, x_2) - x_0^{-1} \delta\left(\frac{-x_2 + x_1}{x_0}\right) Y_W(v, x_2) Y_W(u, x_1) \right] \tag{2.72}$$

$$= \mathrm{Res}_{x_0} \left[x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right)\right] Y_W(u, x_1) Y_W(v, x_2) - \mathrm{Res}_{x_0} \left[x_0^{-1} \delta\left(\frac{-x_2 + x_1}{x_0}\right)\right] Y_W(v, x_2) Y_W(u, x_1) \tag{2.73}$$

$$= Y_W(u, x_1) Y_W(v, x_2) - Y_W(v, x_2) Y_W(u, x_1) = [Y_W(u, x_1), Y_W(v, x_2)] \tag{2.74}$$

and the right-hand side

$$\mathrm{Res}_{x_0}\left[x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_W(Y(u,x_0)v,x_2)\right] = \mathrm{Res}_{x_0}\left[x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\right]Y_W(Y(u,x_0)v,x_2) \tag{2.75}$$

$$= \left[\mathrm{Res}_{x_0}\left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\right) - \mathrm{Res}_{x_0}\left(x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)\right)\right]Y_W(Y(u,x_0)v,x_2) \tag{2.76}$$

$$= (1-1)Y_W(Y(u,x_0)v,x_2) = 0. \tag{2.77}$$

Thus we get $[Y(u,x_1),Y(v,x_2)] = 0$.

Now assume W is a finite dimensional vector space equipped with a linear map $Y_W(\cdot,x) : V \to (\mathrm{End}\ W)[[x,x^{-1}]]$ satisfying

$$\frac{d}{dx}Y_W(v,x) = 0, \tag{2.78}$$

$[Y_W(u,x_1),Y_W(u,x_2)] = 0$, and $Y_W(\mathbf{1},x) = \mathrm{id}_W$ for all $u,v \in V$. Also give W the $\mathbb{C}$-grading defined by $W_{(0)} = W$ and $W_{(h)} = 0$ for $h \neq 0$. Since $L(0) = 0$ and $Y_W(\cdot,x)$ is linear, $L(0)w = 0$, we see that the $W_{(h)}$'s satisfy the definition. Since W is finite dimensional, we also satisfy the grading restriction conditions dim $W_{(h)} < \infty$ for $h \in \mathbb{C}$ and $W_{(h)} = 0$ for h whose real part is sufficiently negative. Since $Y_W(\cdot,x)$ does not depend on the formal variable x, it clearly satisfies the truncation condition, and the vacuum property also holds by hypothesis.

The last property to check is the Jacobi identity. Using $[Y_W(u, x_1), Y_W(v, x_2)] = 0$, we have

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u, x_1)Y_W(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(v, x_2)Y_W(u, x_1) \tag{2.79}$$

$$= x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u, x_1)Y_W(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(u, x_1)Y_W(v, x_2) \tag{2.80}$$

$$= \left[x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\right] Y_W(u, x_1)Y_W(v, x_2) \tag{2.81}$$

$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(u, x_1)Y_W(v, x_2) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(u, x)Y_W(v, x) \tag{2.82}$$

$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(u \cdot v, x) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y(u, x_0)v, x_2), \tag{2.83}$$

showing the Jacobi identity holds. Therefore W is a module of the vertex operator algebra V.

Now we show that the isomorphism classes of modules over the vertex operator algebra V are equivalent to isomorphism classes of modules over V as a commutative associative algebra with identity. Let $\psi : W_1 \to W_2$ be an isomorphism of modules of the vertex operator algebra V. Then defining the action of V on W_1 by

$$v \cdot w \equiv Y_{W_1}(v, x)w \tag{2.84}$$

and the action on W_2 by

$$v \cdot w \equiv Y_{W_2}(v, x)w, \tag{2.85}$$

we see that the condition

$$\psi(v \cdot w + v' \cdot w') = v \cdot \psi(w) + v' \cdot \psi(w') \tag{2.86}$$

is equivalent to

$$\psi(Y_{W_1}(v,x)w + Y_{W_1}(v',x)w') = Y_{W_2}(\psi(w),x) + Y_{W_2}(\psi(w'),x). \tag{2.87}$$

Since ψ is an isomorphism of vertex operator algebra modules, we have

$$\psi(Y_{W_1}(v,x)w + Y_{W_1}(v',x)w') = \psi(Y_{W_1}(v,x)w) + \psi(Y_{W_1}(v',x)w') \tag{2.88}$$

$$= Y_{W_2}(v,x)\psi(w) + Y_{W_2}(v',x)\psi(w') \tag{2.89}$$

by the linearity of ψ and the homomorphism property. Therefore the isomorphism of vertex operator algebra modules is also an isomorphism of their structures as modules of V as a commutative associative algebra with identity. □

2.8 Example: Modules over $V = \mathbb{C}$

Returning to our $V = \mathbb{C}$ example, we will classify the modules of V as a vertex algebra, then V as a vertex operator algebra.

Theorem 2.22 (Classification of $\mathbb{C}$-modules (of the vertex algebra))**.** Any $\mathbb{C}$-module is of the form (W, Y_W) where W is an arbitrary vector space and $Y_W(\cdot, x) : \mathbb{C} \to (\text{End } W)[[x, x^{-1}]]$ sends $u \mapsto u \text{ id}_W$.

Proof. Let (W, Y_W) be a $\mathbb{C}$-module of the vertex algebra $\mathbb{C}$. Since $Y_W(\cdot, x)$ is linear, it is determined by $Y_W(1, x)$. Recall that the vacuum property states that $Y_W(1, x) = \text{id}_W$. Then the map $Y_W(\cdot, x) : \mathbb{C} \to (\text{End } W)[[x, x^{-1}]]$ can only be described by $u \mapsto u \text{ id}_W$. □

Theorem 2.23 (Classification of $\mathbb{C}$-modules (of the vertex operator algebra))**.** Any module of the vertex operator algebra $\mathbb{C}$ is of the form (W, Y_W) where W is an arbitrary finite dimensional vector space and $Y_W(\cdot, x) : \mathbb{C} \to (\text{End } W)[[x, x^{-1}]]$ sends $u \mapsto u \text{ id}_W$.

Proof. We have the grading

$$W = \bigoplus_{h \in \mathbb{C}} W_{(h)} \tag{2.90}$$

where $W_{(0)} = W$ and $W_{(h)} = 0$ for $h \neq 0$. Note that dim $W_{(0)} < \infty$ since W is finite dimensional and dim $W_{(h)} = 0 < \infty$ for $h \neq 0$. In particular, we also have dim $W_{(h)} < \infty$ for h whose real part is sufficiently negative. Therefore all the modules of the vertex algebra $\mathbb{C}$ are also modules of $\mathbb{C}$ as a vertex operator algebra. $\square$

2.9 A remark on the physical interpretation

One would reasonably ask what the point is in using such complicated definitions. A vertex operator algebra provides a space of states together with operators $Y(\cdot, x)$ that allow for the computation of observables in quantum field theory. The world represented by a finite dimensional vertex operator algebra is a vacuum, since an observable of a particle (even a free one) requires an infinite dimensional Hilbert space of states. Naturally, in such a world the central charge is zero, and the creation/annihilation operators $L(n)$ for $n \in \mathbb{Z}$ are zero.

Chapter 3: Conformal blocks and correlation functions of finite dimensional vertex operator algebras

In this chapter I will construct and classify the conformal blocks in the finite dimensional vertex operator algebra case. I will be using the definition described in [FBZ04] throughout the chapter. Note that these results only apply to the case $k = \mathbb{C}$.

3.1 Definition in the general case

In the following construction we use the definitions and similar notation to what is given in [FBZ04]. Let V be a finite dimensional vertex operator algebra. Let $\mathcal{O} \equiv \mathbb{C}[[z]]$ and $\text{Aut } \mathcal{O} \equiv \text{Aut } \mathbb{C}[[z]]$. Let X be a smooth complete (projective) curve and let $x \in X$. We denote the abstract disk at x by $D_x \equiv \text{Spec } \hat{\mathcal{O}}_x$, where $\hat{\mathcal{O}}_x$ is the completed local ring of X at $x \in X$. Let $\mathcal{K}_x$ denote the field of fractions associated with $\hat{\mathcal{O}}_x$, and define the punctured disk $D_x^\times \equiv \text{Spec } \mathcal{K}_x$. Now we will define the notion of coordinates on D_x.

Definition 3.1. A *coordinate* on D_x is a local parameter at x. We then denote by Aut_x the set of coordinates on D_x. Also define

$$Aut_X \equiv \{(x, t_x) | x \in X, t_x \text{ is a local parameter at } x\}. \tag{3.1}$$

Define $\text{Der } \mathcal{O} \equiv \mathbb{C}[[z]]\partial_z$ where ∂_z denotes the partial derivative with respect to z. Next we will define objects $\mathcal{V}_x$ for each point $x \in X$ which are the fibers of the vertex operator algebra bundle $\mathcal{V}$ associated with X and the vertex operator algebra V. Note in the following that $\text{Aut } \mathcal{O}$ has a natural group action on Aut_x, making it an $\text{Aut } \mathcal{O}$-torsor.

Lemma 3.2. The group $\text{Aut } \mathcal{O}$ acts naturally on Aut_x, making it into an $\text{Aut } \mathcal{O}$-torsor.

Proof. We will first prove $\mathcal{O}_x = \hat{\mathcal{O}}_x \simeq \mathbb{C}[[z]]$. Recall that X is a smooth projective curve, and $x \in X$ is a point. The local ring $\mathcal{O}_x$ consists of the regular functions, each in some affine neighborhood of x. By definition such functions are polynomials in a coordinate z in some affine neighborhood around x, implying $\mathcal{O}_x \simeq \mathbb{C}[[z]]$. Now we determine the completed local rings $\hat{\mathcal{O}}_x$. Assign to each $f \in \mathcal{O}_x$ the sequence of residue classes $\xi_n \equiv f + \mathfrak{m}_x^n \in \mathcal{O}_x/\mathfrak{m}_x^n$. Let $\theta_{n+1} : \mathcal{O}_x/\mathfrak{m}_x^{n+1} \to \mathcal{O}_x/\mathfrak{m}_x^n$ be the quotient map. Since $\mathfrak{m}_x = (z)$, we have $\xi_n = f + (z^n)$ and θ_{n+1} is the quotient map $\mathcal{O}_x/(z^{n+1}) \to \mathcal{O}_x/(z^n)$ for all n. Using the method described in [SR94, 103], we see that sequences for any $f \in \mathcal{O}_x$ are precisely the compatible sequences required (that is, satisfying $\theta_{n+1}(\xi_{n+1}) = \xi_n$), and so $\hat{\mathcal{O}}_x = \mathcal{O}_x$ for all $x \in X$. Hence $\hat{\mathcal{O}}_x = \mathbb{C}[[z]]$ for all x.

We then get a right action of Aut $\mathcal{O} \equiv$ Aut $\mathbb{C}[[z]]$ on Aut_x by

$$u \cdot g \equiv g(u) \tag{3.2}$$

for $g \in$ Aut $\mathcal{O}$ and $u \in Aut_x$. Now recall that in order to show Aut_x is an Aut $\mathcal{O}$-torsor under this action, we need to show that given $u, u' \in Aut_x$ there exists a unique element $g \in$ Aut $\mathcal{O}$ such that $u \cdot g = u'$.

It is enough to show that $Aut_x \simeq \mathbb{C} - \{0\}$, as a suitable automorphism g would be uniquely represented by the complex number $g = u'/u$ for $u \neq 0$ (here g is determined by where it sends z by the homomorphism property, and uniqueness would then follow by $g(u)$ being a linear function for $u \in \mathbb{C} - \{0\}$). Recall that local parameters at x are functions $u_1, \ldots, u_n \in \mathcal{O}_x$ that satisfy $u_i \in \mathfrak{m}_x$ for all $1 \leq i \leq n$, and the images of $u_1, \ldots, u_n$ form a basis for $\mathfrak{m}_x/\mathfrak{m}_x^2$ (see [SR94, 98]). Note that $\mathfrak{m}_x/\mathfrak{m}_x^2 = (z)/(z^2) \simeq \mathbb{C}$ as a vector space, so a single local parameter at x can form a basis. Then a local parameter u at x has the form $u = \lambda z \in \mathfrak{m}_x$ with $\lambda \in \mathbb{C}$ and $\lambda \neq 0$, therefore $Aut_x \simeq \mathbb{C} - \{0\}$. □

Definition 3.3. Given our vertex operator algebra V which is also an Aut $\mathcal{O}$-module, define

$\mathcal{V}_x$ to be the Aut_x-twist of V:

$$\mathcal{V}_x \equiv Aut_x \times_{\text{Aut }\mathcal{O}} V. \tag{3.3}$$

Definition 3.4. Define the *vertex operator algebra bundle* over X, denoted $\mathcal{V}$, to be the Aut_X-twist of V:

$$\mathcal{V} \equiv Aut_X \times_{\text{Aut }\mathcal{O}} V. \tag{3.4}$$

Now we will introduce the Lie algebras $U(V)$ and $U(\mathcal{V}_x)$. This content is included for completeness and the reader is encouraged to refer to [FBZ04] as a reference; we do not prove the details in the general case but derive the results in the finite dimensional case and prove the simplified results.

The following definition is intended to adapt the idea of a translation operator used in the definition of a vertex algebra given in [FBZ04]. This is similar to the $L(-1)$-derivative property, and in our more restrictive notion of a vertex operator algebra, the axioms for a vertex algebra used in [FBZ04] are satisfied.

Definition 3.5. Let V be a vertex operator algebra. A *translation operator* is a linear map $T : V \to V$ such that

$$[T, Y(A, z)] = \partial_z Y(A, z) \tag{3.5}$$

and T$\mathbf{1} = 0$.

Now we are ready to define $U(V)$.

Definition 3.6. Consider the linear operator

$$\partial = T \otimes \text{id}_V + \text{id}_V \otimes \partial_t \tag{3.6}$$

on $V \otimes \mathbb{C}((t))$. Then define $U(V)$ to be

$$U(V) \equiv (V \otimes \mathbb{C}((t)))/\text{Im }\partial. \tag{3.7}$$

Remark 3.7. We will describe the Lie algebra structure on $U(V)$ from [FBZ04]. Let

$A \otimes t^n \in V \otimes \mathbb{C}((t))$. Denote its image under the projection

$$V \otimes \mathbb{C}((t)) \to (V \otimes \mathbb{C}((t)))/\text{Im}\ \partial = U(V)$$

by $A_{[n]}$. Now define $[\cdot,\cdot] : U(V) \otimes U(V) \to U(V)$ by

$$[A_{[m]}, B_{[k]}] \equiv \sum_{n \geq 0} \binom{m}{n} (A_n \cdot B)_{[m+k-n]}, \tag{3.8}$$

where $A_n \in \text{End}\ V$ is the nth coefficient in the series

$$Y(A, z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1} \in V[[z, z^{-1}]]. \tag{3.9}$$

We claim this bracket makes $U(V)$ a Lie algebra.

Now to a Zariski open subset $\Sigma \subset X$ we introduce a corresponding Lie algebra $U_\Sigma(V)$ associated with the vertex operator algebra V. In order to do this we first need to define a cohomology theory similar to de Rham cohomology (in fact, this will turn out to be the same in the finite dimensional case).

Remark 3.8. $H^1(\mathcal{V}) \simeq (\mathcal{V} \otimes \Omega)/\text{Im}\ d$, referring to the following complex where $\mu \in \Gamma(D_x^\times, \mathcal{V})$ and $d\mu \in \Gamma(D_x^\times, \mathcal{V} \otimes \Omega)$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{V} & \xrightarrow{d} & \mathcal{V} \otimes \Omega & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \mu} & & \uparrow{\scriptstyle d\mu} & & \\ & & D_x^\times & & D_x^\times & & \end{array}$$

Definition 3.9. Let $\Sigma \subset X$ be a Zariski open subset. Define

$$U_\Sigma(V) \equiv \Gamma(\Sigma, H^1(\mathcal{V})), \tag{3.10}$$

where $H^1(\mathcal{V})$ is the first cohomology of the complex described in Remark 3.8.

Next we define the Lie algebras $U(\mathcal{V}_x)$ and $U_\Sigma(\mathcal{V}_x)$ for $x \in X$ and a Zariski open subset $\Sigma \subset X$.

Definition 3.10. Let $x \in X$ and let $\Sigma \subset X$ be a Zariski open subset. We define

$$U(\mathcal{V}_x) \equiv \Gamma(D_x^\times, H^1(\mathcal{V})). \tag{3.11}$$

There is a homomorphism $U_\Sigma(V) \to U(\mathcal{V}_x)$ induced by the restriction of sections from Σ to $D_x^\times$. Call the image of that homomorphism $U_\Sigma(\mathcal{V}_x)$.

Finally we are ready to define the *space of conformal blocks*.

Definition 3.11. The *space of conformal blocks* associated to (X, x, V) is the space of $U_{X-\{x\}}(\mathcal{V}_x)$-invariant functionals on $\mathcal{V}_x$:

$$C(X, x, V) \equiv \mathrm{Hom}_{U_{X-\{x\}}(\mathcal{V}_x)}(\mathcal{V}_x, \mathbb{C}). \tag{3.12}$$

Remark 3.12. One may wonder why the cohomology $H^1(\mathcal{V})$ is involved in the definition of $U_{X-\{x\}}(\mathcal{V}_x)$ and consequently the space of conformal blocks. This is because the correlation functions can be calculated in terms of a basis of the space of conformal blocks, and since the correlation functions are global objects (keeping track of the curve X), so too are the conformal blocks. The correlation functions are clearly not local since pairs of points can be chosen anywhere on X, and do not have to be restricted to a neighborhood.

The question of why specifically the cohomology theory given by $H^1(\mathcal{V})$ is more challenging to answer. Roughly, the conjectural picture is that this particular cohomology theory is necessary in order to derive the Ward identities (complicated relations the correlation functions of a conformal field theory must satisfy). In particular, we conjecture that in this setting the Ward identities are equivalent to the $U_{X-\{x\}}(\mathcal{V}_x)$-invariant condition in the definition of the space of conformal blocks.

3.2 The finite dimensional case

We note that in the finite dimensional case the definitions of $\mathcal{V}_x$ and $\mathcal{V}$ trivialize, i.e. $\mathcal{V}_x = V$ for all $x \in X$ so that $\mathcal{V}$ becomes the trivial bundle $X \times V$.

Lemma 3.13. If V is a finite dimensional vertex operator algebra, then $\mathcal{V}_x = V$ for every $x \in X$.

Proof. Let $x \in X$. We have by definition

$$\mathcal{V}_x = Aut_x \times_{\text{Aut } \mathcal{O}} V = (Aut_x \times V)/\{(s \cdot g, v) \sim (s, gv) \forall s \in Aut_x, g \in \text{Aut } \mathcal{O}, v \in V\} \tag{3.13}$$

$$= (Aut_x \times V)/\{(s \cdot g, v) \sim (s, v) \forall s \in Aut_x, g \in \text{Aut } \mathcal{O}, v \in V\} \tag{3.14}$$

$$= (Aut_x \times V)/\{(s', v) \sim (s, v) \forall s, s' \in Aut_x\} = V. \tag{3.15}$$

□

Lemma 3.14. If V is a finite dimensional vertex operator algebra, then $\mathcal{V} = X \times V$.

Proof. Similar to the proof of the previous lemma, we have

$$\mathcal{V} = Aut_X \times_{\text{Aut } \mathcal{O}} V = (Aut_X \times V)/\{((x, s) \cdot g, v) \sim ((x, s), gv) \forall (x, s) \in Aut_X, g \in \text{Aut } \mathcal{O}, v \in V\} \tag{3.16}$$

$$= (Aut_X \times V)/\{((x, s) \cdot g, v) \sim ((x, s), v) \forall s \in Aut_X, g \in \text{Aut } \mathcal{O}, v \in V\} \tag{3.17}$$

$$= (Aut_X \times V)/\{((x, s'), v) \sim ((x, s), v) \forall (x, s), (x, s') \in Aut_X\} = X \times V. \tag{3.18}$$

□

Lemma 3.15. Let V be a finite dimensional vertex operator algebra. Then if $T : V \to V$ is a translation operator, $T = 0$.

Proof. Let $v \in V$ be an arbitrary vector and let $T : V \to V$ be a translation operator. Since V is finite dimensional, $\partial_z Y(v,z) = 0$ and so T commutes with $Y(v,z)$. By the creation property we have

$$Tv = T \lim_{z \to 0} Y(v,z)\mathbf{1} = TY(v,z)\mathbf{1} = Y(v,z)T\mathbf{1} = Y(v,z)0 = 0, \tag{3.19}$$

showing that $T = 0$. □

Lemma 3.16. Let V be a finite dimensional vertex operator algebra. Then

$$U(V) = (V \otimes \mathbb{C}((t)))/\mathrm{Im}(\mathrm{id}_V \otimes \partial_t). \tag{3.20}$$

Proof. Follows immediately from $T = 0$. □

We define a Lie bracket on $U(V)$ as follows based on the definition given in the previous section. Note that the notation A_n refers to End V-valued coefficients in the expansion of $Y(A,z)$. See [FBZ04, 44, 63-64], where these are called "Fourier coefficients".

Lemma 3.17. Let $A \otimes t^n \in V \otimes \mathbb{C}((t))$. Denote its image under the projection

$$V \otimes \mathbb{C}((t)) \to (V \otimes \mathbb{C}((t)))/\mathrm{Im}\ \partial = U(V)$$

by $A_{[n]}$. Now define $[\cdot,\cdot] : U(V) \otimes U(V) \to U(V)$ by

$$[A_{[m]}, B_{[k]}] \equiv 0. \tag{3.21}$$

This gives $U(V)$ the structure of a Lie algebra.

Proof. The result immediately follows from the elementary fact that any vector space equipped with the trivial bracket is a Lie algebra. □

Remark 3.18. We will describe where the statement of the above lemma comes from, by deriving the Lie algebra structure on $U(V)$ from what is described in [FBZ04]. Let $A \otimes t^n \in$

$V \otimes \mathbb{C}((t))$. Denote its image under the projection $V \otimes \mathbb{C}((t)) \to (V \otimes \mathbb{C}((t)))/\mathrm{Im}\ \partial = U(V)$ by $A_{[n]}$. Now define $[\cdot, \cdot] : U(V) \otimes U(V) \to U(V)$ by

$$[A_{[m]}, B_{[k]}] \equiv \sum_{n \geq 0} \binom{m}{n} (A_n \cdot B)_{[m+k-n]}, \tag{3.22}$$

where $A_n \in \mathrm{End}\ V$ is the nth coefficient in the series

$$Y(A, z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1} \in V[[z, z^{-1}]]. \tag{3.23}$$

Since V is a finite dimensional vertex operator algebra,

$$\frac{d}{dz} Y(A, z) = 0, \tag{3.24}$$

and so $Y(A, z) = A_{-1} \in \mathrm{End}\ V$. We then have

$$[A_{[m]}, B_{[k]}] = \sum_{n \geq 0} \binom{m}{n} (A_n \cdot B)_{[m+k-n]} = \sum_{n \geq 0} \binom{m}{n} (0 \cdot B)_{[m+k-n]} \tag{3.25}$$

$$= \sum_{n \geq 0} \binom{m}{k} 0_{[m+k-n]} = 0, \tag{3.26}$$

showing that $U(V)$ is equipped with the trivial bracket.

Theorem 3.19. Let V be a finite dimensional vertex operator algebra. Let t be a coordinate at $x \in X$. Then

$$U(\mathcal{V}_x) \simeq U(V) = (V \otimes \mathbb{C}((t)))/\mathrm{Im}(\mathrm{id}_V \otimes \partial_t). \tag{3.27}$$

Proof. Let $d : \mathcal{V} \to \mathcal{V} \otimes \Omega$ be the boundary map from de Rham cohomology. Let t be a coordinate at $x \in X$. We want to show that

$$\Gamma(D_x^\times, H^1_{dR}(\mathcal{V})) \simeq V \otimes \mathbb{C}((t))/\mathrm{Im}(\mathrm{id}_V \otimes \partial_t). \tag{3.28}$$

Consider the complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{V} & \xrightarrow{d} & \mathcal{V} \otimes \Omega & \longrightarrow & 0 \\ & & \mu \big\uparrow & & d\mu \big\uparrow & & \\ & & D_x^\times & & D_x^\times & & \end{array}$$

where $\mu \in \Gamma(D_x^\times, \mathcal{V})$ and $d\mu \in \Gamma(D_x^\times, \mathcal{V} \otimes \Omega)$ are sections of the bundle $\mathcal{V}$ over X. Note that Im d consists of the exact forms on $\mathcal{V}$ by definition. Consider the quotient map

$$\phi : \mathcal{V} \otimes \Omega \to (\mathcal{V} \otimes \Omega)/\mathrm{Im}\ d.$$

Since $d^2(\rho) = 0$ for $\rho \in \mathcal{V}$, we see that every element of $\mathcal{V} \otimes \Omega$ is a closed form, hence we make the identification $(\mathcal{V} \otimes \Omega)/\mathrm{Im}\ d \simeq H^1_{dR}(\mathcal{V})$.

Now we write $\mu(t)$ and $d\mu(t)$ in terms of their factors. We write $\mu(t) = (A(t), B(t)) \in \mathcal{V} = X \times \mathcal{V}$. Let $\pi : \mathcal{V} \to X$ denote the projection map for the bundle $\mathcal{V}$ over X. By definition of a section over a bundle, $\mu : D_x^\times \to \mathcal{V}$ satisfies $\pi(\mu(t)) = t$ for all $t \in D_x^\times$ and is continuous. That is, $\pi(\mu(t)) = \pi(A(t), B(t)) = A(t) = t$, and so μ has the form $\mu(t) = (t, B(t))$. Note that after choosing a coordinate t at x, we have $d = \partial_t \otimes dt$. It then follows that $d\mu(t) = (1, \partial_t B(t)) \otimes dt \in \mathcal{V} \otimes \Omega$.

We also have the map into cohomology given by

$$\Gamma(D_x^\times, \mathcal{V}) \to \Gamma(D_x^\times, H^1_{dR}(\mathcal{V})) \tag{3.29}$$

$$\mu \mapsto [\mu] = \mu + d(\Gamma(D_x^\times, \mathcal{V})). \tag{3.30}$$

Now we use this description of the sections to write down a map $U(\mathcal{V}_x) \to U(V)$. Since the definition of a section forced $A(t) = t$, intuitively the X factors out of $U(\mathcal{V}_x)$. Define the map

$$\rho : \Gamma(D_x^\times, H^1_{dR}(\mathcal{V})) \to V \otimes \mathbb{C}((t))/\mathrm{Im}(\mathrm{id}_V \otimes \partial_t) \tag{3.31}$$

$$\mu \mapsto [B(t) \otimes 1] = B. \tag{3.32}$$

One can see that it's clearly an isomorphism. The factor $\mathbb{C}((t))$ indeed must be truncated in order to ensure the continuity of the sections, since we are working over the punctured disk $D_x^\times$. □

Remark 3.20. The argument in the proof above for X to factor out can be extended to the case where $\mathcal{V}$ isn't a trivial bundle. This is since the sections $\mu \in \Gamma(D_x^\times, H^1_{dR}(\mathcal{V}))$ are local, and so the bundle need only be locally trivial for the argument to stand. Fortunately, $\mathcal{V}$ is a locally trivial bundle in general, although here we only consider the case where V is a finite dimensional vertex operator algebra and as a result $\mathcal{V}$ is the trivial bundle $X \times V$.

Remark 3.21. The "de Rham cohomology" from [FBZ04] is not the literal de Rham cohomology in general, since the boundary map has an extra term when V is not commutative. In the general case, define the cohomology using the abstract definition with the provided flat connection and complex. Note that in the case where V is a finite dimensional vertex operator algebra, the cohomology is the literal de Rham cohomology since the $L(-1)$ term vanishes but the d remains in the expression for the connection.

Having shown the identification $U(\mathcal{V}_x) \simeq U(V)$, we can induce a Lie algebra structure on $U(\mathcal{V}_x)$ from the Lie algebra structure on $U(V)$. This is trivial in the finite dimensional case since $U(V)$ is a trivial Lie algebra.

Lemma 3.22. $U(\mathcal{V}_x)$ is a Lie algebra with the trivial bracket.

Proof. As in the result for $U(V)$, this follows immediately from the fact that any vector space equipped with the trivial bracket is a Lie algebra. □

Now we can get a Lie algebra structure on $U_\Sigma(V)$ where $\Sigma \subset X$ is a Zariski open subset.

Lemma 3.23. Let $\Sigma \subset X$ be a Zariski open subset. $U_\Sigma(V)$ is a Lie algebra with the trivial bracket.

Proof. Same as before; $U_\Sigma(V)$ is a vector space and so can be given the trivial bracket to make it a Lie algebra. $\square$

Remark 3.24. Let V be a finite dimensional vertex operator algebra. Recall from [FBZ04, 156-157] that the restriction of sections from a Zariski open subset $\Sigma \subset X$ to $D_x^\times$ for a suitable point $x \in X$ yields a Lie algebra homomorphism $U_\Sigma(V) \to U(\mathcal{V}_x)$. Thus we give $U_\Sigma(V)$ the trivial bracket since Σ could be the punctured disk $D_x^\times$, in which case the trivial bracket is necessary to make the map a Lie algebra homomorphism by virtue of $U(\mathcal{V}_x)$ having the trivial bracket.

Now we construct the last Lie algebra $U_\Sigma(\mathcal{V}_x)$ that we are interested in for the case where V is a finite dimensional vertex operator algebra.

Lemma 3.25. Let $U_\Sigma(\mathcal{V}_x)$ be the image of the Lie algebra homomorphism $U_\Sigma(V) \to U(\mathcal{V}_x)$. Then this induces a Lie algebra structure on $U_\Sigma(\mathcal{V}_x)$ with the trivial bracket.

Proof. Recall that the image of a Lie algebra homomorphism is a Lie subalgebra of the codomain. In this case, $U_\Sigma(\mathcal{V}_x)$ is a Lie subalgebra of $U(\mathcal{V}_x)$. As a Lie subalgebra, $U_\Sigma(\mathcal{V}_x)$ inherits the Lie bracket defined on $U(\mathcal{V}_x)$, which is the trivial bracket. $\square$

We will construct a linear function $\mathcal{Y}_x : \Gamma(D_x^\times, \mathcal{V}^*) \to \text{End } \mathcal{V}_x$, then a dual map $\mathcal{Y}_x^\vee : \Gamma(D_x^\times, \mathcal{V} \otimes \Omega) \to \text{End } \mathcal{V}_x$, and finally a map $\phi : U_\Sigma(V) \to U(\mathcal{V}_x)$. The composition $\mathcal{Y}_x^\vee \circ \phi$ will then give us an action of $U_\Sigma(V)$ on $\mathcal{V}_x$.

Remark 3.26. Choose a coordinate t_x on the disk D_x. Then we have the isomorphisms $\hat{\mathcal{O}}_x \simeq \mathbb{C}[[t_x]]$ and $\mathcal{K}_x \simeq \mathbb{C}((t_x))$. By making these identifications we will be able to make sense of the map $\pi' : \text{Spec } \hat{\mathcal{O}}_x \to \text{Spec } \mathcal{K}_x$ as a restriction.

Remark 3.27. We obtain the map ϕ by restriction of sections from Σ to $D_x^\times$. In the case we are interested in, $\Sigma = X - \{x\}$. The punctured disk $D_x^\times$ is thought of as missing the origin x, so it is contained in Σ. We have the following diagram showing the involved maps.

$$\begin{array}{ccc} \Sigma & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\phi'=\pi'\circ\pi} & \\ \text{Spec } \hat{\mathcal{O}}_x & \xrightarrow{\pi'} & \text{Spec } \mathcal{K}_x \end{array}$$

Recall that the $\mathcal{K}_x = \hat{\mathcal{O}}_x/\hat{\mathfrak{m}}_x$ is the field of fractions associated with the complete local ring $\hat{\mathcal{O}}_x$, where $\hat{\mathfrak{m}}_x$ is the associated maximal ideal. The map $\pi : \text{Spec } \hat{\mathcal{O}}_x \to \text{Spec } \mathcal{K}_x$ is induced by the map $\hat{\mathcal{O}}_x \to \hat{\mathcal{O}}_x/\hat{\mathfrak{m}}_x = \mathcal{K}_x$ defined by $u \mapsto u + \hat{\mathfrak{m}}_x$. We then induce the map $\phi : \Gamma(\Sigma, H^1_{dR}(\mathcal{V})) \to \Gamma(D_x^\times, H^1_{dR}(\mathcal{V}))$ from ϕ' in the above diagram. More concretely, ϕ is defined by $\mu \mapsto \mu|_{D_x^\times} = \mu \circ (\phi')^{-1}$.

$\mathcal{Y}_x$ is defined by an equation given in Theorem/Definition 6.5.4 of [FBZ04, 114].

Definition 3.28. Let $\mu \in \Gamma(D_x^\times, \mathcal{V}^*)$ and $\psi \in V^*$. Let z be a formal coordinate at x. Then $\mathcal{Y}_x : \Gamma(D_x^\times, \mathcal{V}^*) \to \text{End } \mathcal{V}_x$ is defined by

$$\tilde{\psi}(\mathcal{Y}_x(\mu)\tilde{v}) = \psi(Y(A, z)v) \tag{3.33}$$

where $\tilde{\psi} = (z, \psi) \in \mathcal{V}^*$ and $\tilde{v} = (z, v) \in \mathcal{V}$.

We get the following result in our special case.

Lemma 3.29. Let $\mu \in \Gamma(D_x^\times, \mathcal{V}^*)$ be arbitrary. $\mathcal{Y}_x(\mu)$ is the map $\mathcal{V}_x \to V$ defined by $Y(A, z)$, that is

$$\mathcal{Y}_x(\mu) \cdot (z, v) = Y(A, z)v. \tag{3.34}$$

Proof. Since V is finite dimensional (say dimension $d \geq 0$), we can represent the element $\psi \in V^*$ by a $d \times d$ matrix $\lambda \in M_{d\times d}(\mathbb{C})$. Then by linearity of the endomorphism $\mathcal{Y}_x(\mu)$ of $\mathcal{V}_x$, we have

$$\lambda\mathcal{Y}_x(\mu) \cdot (z, v) = \lambda Y(A, z)v. \tag{3.35}$$

Recall that in the finite dimensional case the map $Y(A,z) \in \text{End } V$ (i.e. does not depend on the formal variable z). Since the equation is satisfied for all $\lambda \in M_{2\times 2}(\mathbb{C})$, we can set λ to the $d \times d$ identity matrix. Then

$$\mathcal{Y}_x(\mu) \cdot (z, v) = Y(A, z)v. \tag{3.36}$$

From the above equation it immediately follows that $\mathcal{Y}_x(\mu)$ acts as the linear map $Y(A,z)$. □

We will need the following alternate use of *residue map*.

Definition 3.30. Let Ω_x denote the module of differentials on the punctured disk $D_x^{\times}$. Let $\omega_x \in \Omega_x$ be a one-form given by the expression

$$\omega_x = \sum_{i \in \mathbb{Z}} f_i t^i dt. \tag{3.37}$$

We define its *residue* as

$$\text{Res}_x \omega_x = f_{-1}. \tag{3.38}$$

Next is the definition for $\mathcal{Y}_x^{\vee}$.

Definition 3.31. Define $\mathcal{Y}_x^{\vee} : U(\mathcal{V}_x) \to \text{End } \mathcal{V}_x$ by

$$\mu \mapsto O_\mu = \text{Res}_x \mathcal{Y}_x(\mu). \tag{3.39}$$

Now we are ready to work out $\mathcal{Y}_x^{\vee}$ in our particular case. We have the following result.

Lemma 3.32. Let $\mu = A \otimes z^n dz \in \Gamma(D_x^{\times}, \mathcal{V}^*)$ be arbitrary. Then $\mathcal{Y}_x^{\vee}(\mu) = 0$.

Proof. By definition we have

$$\mathcal{Y}_x^{\vee}(\mu) = \text{Res}_x \mathcal{Y}_x(\mu) = 0, \tag{3.40}$$

since the endomorphism $\mathcal{Y}_x(\mu)$ does not depend on z. ☐

Now we are ready to prove our result.

Theorem 3.33. The space of conformal blocks is given by

$$C(X, x, V) = \mathcal{F}(V, \mathbb{C}), \tag{3.41}$$

where $\mathcal{F}(V, \mathbb{C})$ denotes the one dimensional space of constant functions $V \to \mathbb{C}$.

Proof. Recall that $U_{X-\{x\}}(\mathcal{V}_x)$ acts on $\mathcal{V}_x$ via the composition $\mathcal{Y}_x^\vee \circ \phi$ which gives an endomorphism of $\mathcal{V}_x$. Since $\mathcal{Y}_x^\vee$ is the zero function, the action sends every element of $\mathcal{V}_x$ to zero. Recall the definition of the space of conformal blocks,

$$C(X, x, V) = \mathrm{Hom}_{U_{X-\{x\}}(\mathcal{V}_x)}(\mathcal{V}_x, \mathbb{C}). \tag{3.42}$$

Elements of this space are $U_{X-\{x\}}(\mathcal{V}_x)$-invariant functionals, i.e. a functional $\tau : \mathcal{V}_x \to \mathbb{C}$ is an element of $C(X, x, V)$ if and only if

$$\rho \cdot \tau(\tilde{v}) = \tau(\tilde{v}) \tag{3.43}$$

for all $\rho \in U_{X-\{x\}}(\mathcal{V}_x)$ and $\tilde{v} \in \mathcal{V}_x$. This is equivalent to the condition

$$\tau((\mathcal{Y}_x^\vee \circ \phi)(\tilde{v})) = \tau(\tilde{v}), \tag{3.44}$$

which is equivalent to the condition $\tau(0) = \tau(\tilde{v})$ for all $\tilde{v} \in \mathcal{V}_x$ since $\mathcal{Y}_x^\vee$ is the zero function. Therefore the elements of $C(X, x, V)$ are precisely the constant functions $\mathcal{V}_x \to \mathbb{C}$, and so

$$C(X, x, V) = \mathcal{F}(V, \mathbb{C}). \tag{3.45}$$

☐

3.3 N-point conformal blocks with module insertions

This section will follow the definitions given in Chapter 10 of [FBZ04]. The following is an analogue of the space of conformal blocks defined previously for multiple points. Note that the notion of a "vertex operator algebra module bundle" simply involves applying the construction of a vertex operator algebra bundle given in Section 3.1 to modules. We get analogous results for the modules in the finite dimensional vertex operator algebra case since the maps $Y_M(\cdot, z)$ don't depend on the formal variable z, as shown in Chapter 2.

Remark 3.34. We will extend the action of $U_\Sigma(V)$ on V given by the map $\mathcal{Y}_x^\vee$ to an action of $U_\Sigma(V)$ on a tensor product of vertex operator algebra module bundles. We obtain analogues of the map $\mathcal{Y}_x^\vee$ for each module bundle $\mathcal{M}_{i,x_i}$, denoted by $\mathcal{Y}_{M_i,x_i}^\vee : U(\mathcal{V}_{x_i}) \to \operatorname{End} \mathcal{M}_{i,x_i}$. We then obtain the action using

$$\mathcal{Y}_{M,x_1}^\vee \otimes \cdots \otimes \mathcal{Y}_{M,x_N}^\vee : \bigoplus_{i=1}^N U(\mathcal{V}_{x_i}) \to \bigotimes_{i=1}^N \mathcal{M}_{i,x_i} \tag{3.46}$$

and a map

$$U_\Sigma(V) \to \bigoplus_{i=1}^N U(\mathcal{V}_{x_i}) \tag{3.47}$$

induced by the tuple of restrictions of sections

$$\Gamma(\Sigma, H^1(\mathcal{V})) \to \bigoplus_{i=1}^N \Gamma(D_{x_i}^\times, H^1(\mathcal{V})). \tag{3.48}$$

Definition 3.35. Let V be a vertex operator algebra, X be a smooth projective curve, and $\Sigma \equiv X - \{x_1, \ldots, x_N\}$ a Zariski open subset of X. Let $M_1, \ldots, M_N$ be modules of the vertex operator algebra V. Let $\mathcal{M}_1, \ldots, \mathcal{M}_N$ denote the associated vertex operator algebra module bundles, and $\mathcal{M}_{1,x_1}, \ldots, \mathcal{M}_{N,x_N}$ their corresponding fibers. Then we define the associated *space of conformal blocks* to be

$$C_V(X,(x_i),(M_i))_{i=1}^N \equiv \mathrm{Hom}_{U_\Sigma(V)}\left(\bigotimes_{i=1}^N \mathcal{M}_{i,x_i}, \mathbb{C}\right). \tag{3.49}$$

Remark 3.36. The above definition in the case $N=1$ where M is the adjoint module of V is the notion of the space of conformal blocks considered in previous sections.

We have the following analogous result for finite dimensional vertex operator algebras based on the argument for the simpler case shown before.

Theorem 3.37. Let V be a finite dimensional vertex operator algebra. The space of conformal blocks is given by

$$C_V(X,(x_i),(M_i))_{i=1}^N = \mathcal{F}\left(\bigotimes_{i=1}^N M_i, \mathbb{C}\right), \tag{3.50}$$

where as before $\mathcal{F}\left(\bigotimes_{i=1}^N M_i, \mathbb{C}\right)$ denotes the space of constant functions $\bigotimes_{i=1}^N M_i \to \mathbb{C}$.

Remark 3.38. Note that in the above result the space of conformal blocks does not depend on X or the choice of points (x_i), since the bundles $\mathcal{M}_i = X \times M_i$ are trivial as in the simpler case with a single point.

3.4 Correlation functions

Other objects of interest are the correlations functions.

Definition 3.39. Let V be a vertex operator algebra. Let $A_1, \ldots, A_n \in V$. For any $\phi \in V^*$, and any permutation σ of n elements, the formal power series of the form

$$\langle \phi, Y(A_{\sigma(1)}, z_{\sigma(1)}) \cdots Y(A_{\sigma(n)}, z_{\sigma(n)})\mathbf{1}\rangle \tag{3.51}$$

are the *correlation functions*.

We have the following simplification in the finite dimensional case.

Theorem 3.40. Let V be a finite dimensional vertex operator algebra. Then the correlation functions have the form

$$\phi(A_{\sigma(1)} \cdots A_{\sigma(n)}), \tag{3.52}$$

where $\phi \in V^*$, σ is a permutation of n elements, and $A_1, \ldots, A_n \in V$. The product of the A_i's is the product of V as a commutative associative algebra.

Proof. We directly compute an arbitrary correlation function as

$$\langle \phi, Y(A_{\sigma(1)}, z_{\sigma(1)}) \cdots Y(A_{\sigma(n)}, z_{\sigma(n)})\mathbf{1} \rangle = \langle \phi, A_{\sigma(1)} \cdots A_{\sigma(n)}\mathbf{1} \rangle \tag{3.53}$$

$$= \phi(A_{\sigma(1)} \cdots A_{\sigma(n)}). \tag{3.54}$$

□

Remark 3.41. Note that in the finite dimensional case, elements of the form

$$Y(A_{\sigma(1)}, z_{\sigma(1)}) \cdots Y(A_{\sigma(n)}, z_{\sigma(n)})\mathbf{1} \tag{3.55}$$

generate V, and so the correlation functions $\langle \phi, \cdot \rangle$ are the constant functions $V \to \mathbb{C}$.

Chapter 4: Review of infinite dimensional Lie algebras

The purpose of this chapter is to review the material concerning Lie algebras necessary to fill in the relevant details to the construction of the vertex operator algebra associated with the $\mathfrak{sl}_2(\mathbb{C})$ WZW model in the non-critical case.

4.1 The simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

We recall that $\mathfrak{sl}_2(\mathbb{C})$ is the set of traceless 2×2 matrices with complex entries. The following is a review of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$; a good reference for the material covered here is [H+03]. As a vector space $\mathfrak{sl}_2(\mathbb{C})$ has a basis

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{4.1}$$

that we will be using throughout this section.

We will use the following definition of a finite dimensional real or complex Lie algebra from [H+03] and show that $\mathfrak{sl}_2(\mathbb{C})$ is a Lie algebra.

Definition 4.1. A finite dimensional real or complex Lie algebra is a finite dimensional real or complex vector space $\mathfrak{g}$, together with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that (1) $[\cdot,\cdot]$ is bilinear, (2) $[\cdot,\cdot]$ is skew symmetric ($[X,Y] = -[Y,X]$ for all $X, Y \in \mathfrak{g}$), and (3) the Jacobi identity holds:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{4.2}$$

for all $X, Y, Z \in \mathfrak{g}$.

Theorem 4.2. $\mathfrak{sl}_2(\mathbb{C})$ is a complex Lie algebra with the bracket $[X,Y] \equiv XY - YX$ for all $X, Y \in \mathfrak{sl}_2(\mathbb{C})$.

Proof. First we check that the bracket is bilinear. Let $\lambda, \lambda' \in \mathbb{C}$. Then

$$[\lambda X + \lambda' Y, Z] = (\lambda X + \lambda' Y)Z - Z(\lambda X + \lambda' Y) = \lambda XZ + \lambda' YZ - \lambda ZX - \lambda' ZY \quad (4.3)$$

$$= \lambda(XZ - ZX) + \lambda'(YZ - ZY) = \lambda[X,Z] + \lambda'[Y,Z] \quad (4.4)$$

and

$$[Z, \lambda X + \lambda' Y] = Z(\lambda X + \lambda' Y) - (\lambda X + \lambda' Y)Z = \lambda ZX + \lambda' ZY - \lambda XZ - \lambda' YZ \quad (4.5)$$

$$= \lambda(ZX - XZ) + \lambda'(ZY - YZ) = \lambda[Z,X] + \lambda'[Z,Y] \quad (4.6)$$

for every $X, Y, Z \in \mathfrak{sl}_2(\mathbb{C})$. Nest we check that the bracket is skew symmetric. We have

$$[X,Y] = XY - YX = -(-XY + YX) = -(YX - XY) = -[Y,X] \quad (4.7)$$

for all $X, Y \in \mathfrak{sl}_2(\mathbb{C})$. Lastly, we check that the Jacobi identity holds. We have

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = [X, YZ - ZY] + [Y, ZX - XZ] + [Z, XY - YX] \quad (4.8)$$

$$= [X, YZ] - [X, ZY] + [Y, ZX] - [Y, XZ] + [Z, XY] - [Z, YX] \quad (4.9)$$

$$= (XYZ - YZX) - (XZY - ZYX) + (YZX - ZXY) + (ZXY - XYZ) - (ZYX - YXZ) \quad (4.10)$$

$$= XYZ - YZX - XZY + ZYX + YZX - ZXY + ZXY - XYZ - ZYX + YXZ = 0. \quad (4.11)$$

Therefore $\mathfrak{sl}_2(\mathbb{C})$ is a Lie algebra. □

Note that the above proof generalizes to any Lie algebra $\mathfrak{g}$ where the bracket is defined by $[X,Y] \equiv XY - YX$ for all $X, Y \in \mathfrak{g}$.

4.2 The affine Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$

Set theoretically, we define $\hat{\mathfrak{sl}}_2(\mathbb{C})$ as

$$\hat{\mathfrak{sl}}_2(\mathbb{C}) \equiv (\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}k \tag{4.12}$$

where k is a nonzero central element of $\hat{\mathfrak{sl}}_2(\mathbb{C})$. The part in the parentheses is the loop algebra associated with $\mathfrak{sl}_2(\mathbb{C})$:

$$\mathcal{L}(\mathfrak{sl}_2(\mathbb{C})) \equiv \mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]. \tag{4.13}$$

One can show that $\hat{\mathfrak{sl}}_2(\mathbb{C})$ is an infinite dimensional complex Lie algebra with a suitable choice of bracket. We have the following more general definition than in the previous example of a Lie algebra.

Definition 4.3. A Lie algebra is a vector space $\mathfrak{g}$ over a field F with a binary linear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following conditions: (1) $[\cdot,\cdot]$ is bilinear, i.e. $[ax+by, z] = a[x,z] + b[y,z]$ and $[z, ax+by] = a[z,x] + b[z,y]$ for all $a, b \in F$ and $x, y, z \in \mathfrak{g}$, (2) $[x,x] = 0$ for every $x \in \mathfrak{g}$, and (3) the Jacobi identity holds:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \tag{4.14}$$

for all $x, y, z \in \mathfrak{g}$.

Next we will show that $\hat{\mathfrak{sl}}_2(\mathbb{C})$ is a Lie algebra.

Theorem 4.4. $\hat{\mathfrak{sl}}_2(\mathbb{C})$ is a Lie algebra with the bracket

$$[a \otimes t^m, b \otimes t^n] \equiv [a, b] \otimes t^{m+n} + m\mathrm{Tr}(ab)\delta_{m+n,0}k \tag{4.15}$$

defined in terms of the bracket for the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ for all $a, b \in \mathfrak{sl}_2(\mathbb{C})$ and $m, n \in \mathbb{Z}$.

Proof. First we check bilinearity. Let $\lambda, \lambda' \in \mathbb{C}$ and $a \otimes t^p, b \otimes t^q, d \otimes t^r \in \hat{\mathfrak{sl}}_2(\mathbb{C})$. Then

$$[\lambda(a \otimes t^p) + \lambda'(b \otimes t^q), d \otimes t^r] = [\lambda(a \otimes t^p), d \otimes t^r] + [\lambda'(b \otimes t^q), d \otimes t^r] \tag{4.16}$$

$$= ([\lambda a, d] \otimes t^{p+r} + p\mathrm{Tr}(\lambda ad)\delta_{p+r,0}k) + ([\lambda' b, d] \otimes t^{q+r} + q\mathrm{Tr}(\lambda' bd)\delta_{q+r,0}k) \tag{4.17}$$

$$= \lambda([a, d] \otimes t^{p+r} + p\mathrm{Tr}(ad)\delta_{p+r,0}k) + \lambda'([b, d] \otimes t^{q+r} + q\mathrm{Tr}(bd)\delta_{q+r,0}k) \tag{4.18}$$

$$= \lambda[a \otimes t^p, d \otimes t^r] + \lambda'[b \otimes t^q, d \otimes t^r] \tag{4.19}$$

and

$$[d \otimes t^r, \lambda(a \otimes t^p) + \lambda'(b \otimes t^q)] = [d \otimes t^r, \lambda(a \otimes t^p)] + [d \otimes t^r, \lambda'(b \otimes t^q)] \tag{4.20}$$

$$= ([d, \lambda a] \otimes t^{r+p} + r\mathrm{Tr}(d(\lambda a))\delta_{r+p,0}k) + ([d, \lambda' b] \otimes t^{r+q} + r\mathrm{Tr}(d(\lambda' b))\delta_{r+q,0}k) \tag{4.21}$$

$$= \lambda([d, a] \otimes t^{r+p} + r\mathrm{Tr}(da)\delta_{r+p,0}k) + \lambda'([d, b] \otimes t^{r+q} + r\mathrm{Tr}(db)\delta_{r+q,0}k) \tag{4.22}$$

$$= \lambda[d \otimes t^r, a \otimes t^p] + \lambda'[d \otimes t^r, b \otimes t^q]. \tag{4.23}$$

Note that in the above we interpret the definition of the bracket for non-pure tensors by extending linearly. We also have

$$[a \otimes t^m, a \otimes t^m] = [a, a] \otimes t^{m+m} + m\mathrm{Tr}(aa)\delta_{m+m,0}k \tag{4.24}$$

$$= 0 \otimes t^{2m} + m\mathrm{Tr}(a^2)\delta_{2m,0}k = 0. \tag{4.25}$$

Lastly, we show that the Jacobi identity holds for $a \otimes t^p, b \otimes t^q, d \otimes t^r \in \hat{\mathfrak{sl}}_2(\mathbb{C})$. We have

$$[a \otimes t^p, [b \otimes t^q, d \otimes t^r]] + [b \otimes t^q, [d \otimes t^r, a \otimes t^p]] + [d \otimes t^r, [a \otimes t^p, b \otimes t^q]] \tag{4.26}$$

$$= [a \otimes t^p, [b, d] \otimes t^{q+r} + q\mathrm{Tr}(bd)\delta_{q+r,0}k] + [b \otimes t^q, [d, a] \otimes t^{r+p} + r\mathrm{Tr}(da)\delta_{r+p,0}k] \tag{4.27}$$

$$+ [d \otimes t^r, [a, b] \otimes t^{p+q} + p\mathrm{Tr}(ab)\delta_{p+q,0}k] \tag{4.28}$$

$$= ([a \otimes t^p, [b, d] \otimes t^{q+r}] + [a \otimes t^p, q\mathrm{Tr}(bd)\delta_{q+r,0}k]) \tag{4.29}$$

$$+ ([b \otimes t^q, [d, a] \otimes t^{r+p}] + [b \otimes t^q, r\mathrm{Tr}(da)\delta_{r+p,0}k]) \tag{4.30}$$

$$+ ([d \otimes t^r, [a, b] \otimes t^{p+q}] + [d \otimes t^r, p\mathrm{Tr}(ab)\delta_{p+q,0}k]). \tag{4.31}$$

Now we evaluate each of the terms:

$$[a \otimes t^p, [b, d] \otimes t^{q+r}] = [a, [b, d]] \otimes t^{p+q+r} + p\mathrm{Tr}(a[b, d])\delta_{p+q+r,0}k, \tag{4.32}$$

$$[b \otimes t^q, [d, a] \otimes t^{r+p}] = [b, [d, a]] \otimes t^{p+q+r} + q\mathrm{Tr}(b[d, a])\delta_{p+q+r,0}k, \tag{4.33}$$

$$[d \otimes t^r, [a, b] \otimes t^{p+q}] = [d, [a, b]] \otimes t^{p+q+r} + r\mathrm{Tr}(d[a, b])\delta_{p+q+r,0}k, \tag{4.34}$$

$$[a \otimes t^p, q\mathrm{Tr}(bd)\delta_{q+r,0}k] = q\mathrm{Tr}(bd)\delta_{q+r,0}[a \otimes t^p, k] = 0, \tag{4.35}$$

$$[b \otimes t^q, r\mathrm{Tr}(da)\delta_{r+p,0}k] = r\mathrm{Tr}(da)\delta_{r+p,0}[b \otimes t^q, k] = 0, \tag{4.36}$$

$$[d \otimes t^r, p\mathrm{Tr}(ab)\delta_{p+q,0}k] = p\mathrm{Tr}(ab)\delta_{p+q,0}[d \otimes t^r, k] = 0. \tag{4.37}$$

Substituting these into the Jacobi identity results in

$$[a \otimes t^p, [b \otimes t^q, d \otimes t^r]] + [b \otimes t^q, [d \otimes t^r, a \otimes t^p]] + [d \otimes t^r, [a \otimes t^p, b \otimes t^q]] \tag{4.38}$$

$$= [a, [b, d]] \otimes t^{p+q+r} + p\mathrm{Tr}(a[b, d])\delta_{p+q+r,0}k + [b, [d, a]] \otimes t^{p+q+r} \tag{4.39}$$

$$+ q\mathrm{Tr}(b[d, a])\delta_{p+q+r,0}k + [d, [a, b]] \otimes t^{p+q+r} + r\mathrm{Tr}(d[a, b])\delta_{p+q+r,0}k \tag{4.40}$$

$$= ([a, [b, d]] + [b, [d, a]] + [d, [a, b]]) \otimes t^{p+q+r} \tag{4.41}$$

$$+ (p\mathrm{Tr}(a[b, d]) + q\mathrm{Tr}(b[d, a]) + r\mathrm{Tr}(d[a, b]))\delta_{p+q+r,0}k \tag{4.42}$$

$$= 0 \otimes t^{p+q+r} + (p\mathrm{Tr}(a[b, d]) + q\mathrm{Tr}(b[d, a]) + r\mathrm{Tr}(d[a, b]))\delta_{p+q+r,0}k \tag{4.43}$$

$$= (p\mathrm{Tr}(a[b, d]) + q\mathrm{Tr}(b[d, a]) + r\mathrm{Tr}(d[a, b]))\delta_{p+q+r,0}k \tag{4.44}$$

$$= (p\mathrm{Tr}(a(bd - db)) + q\mathrm{Tr}(b(da - ad)) + r\mathrm{Tr}(d(ab - ba)))\delta_{p+q+r,0}k \tag{4.45}$$

$$= (p\mathrm{Tr}(abd) - p\mathrm{Tr}(adb) + q\mathrm{Tr}(bda) - q\mathrm{Tr}(bad) + r\mathrm{Tr}(dab) - r\mathrm{Tr}(dba))\delta_{p+q+r,0}k \tag{4.46}$$

$$= (p\mathrm{Tr}(abd) - p\mathrm{Tr}(adb) + q\mathrm{Tr}(abd) - q\mathrm{Tr}(adb) + r\mathrm{Tr}(abd) - r\mathrm{Tr}(adb))\delta_{p+q+r,0}k \tag{4.47}$$

$$= ((p + q + r)\mathrm{Tr}(abd) - (p + q + r)\mathrm{Tr}(adb))\delta_{p+q+r,0}k = 0. \tag{4.48}$$

Therefore $\hat{\mathfrak{sl}}_2(\mathbb{C})$ is a Lie algebra. □

We also define a class of $\hat{\mathfrak{sl}}_2(\mathbb{C})$-modules called the restricted $\hat{\mathfrak{sl}}_2(\mathbb{C})$-modules.

Definition 4.5. Let W be a $\hat{\mathfrak{sl}}_2(\mathbb{C})$-module, and let $a(n)$ denote the operator on W corresponding to $a \otimes t^n \in \hat{\mathfrak{sl}}_2(\mathbb{C})$ for each $a \in \mathfrak{sl}_2(\mathbb{C})$ and $n \in \mathbb{Z}$. W is *restricted* if for every $a \in \mathfrak{sl}_2(\mathbb{C})$ and $w \in W$, we have $a(n)w = 0$ for n sufficiently large.

4.3 The Virasoro algebra

As a vector space, we define the *Virasoro algebra* to be

$$\mathcal{L} \equiv \operatorname{span}_{\mathbb{C}}\{c, L_n | n \in \mathbb{Z}\}. \tag{4.49}$$

Now we will show $\mathcal{L}$ is a Lie algebra.

Theorem 4.6. $\mathcal{L}$ is a Lie algebra with the bracket $[\cdot,\cdot] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ defined by $[c, L_n] = 0$ and

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{4.50}$$

Proof. Note that we extend the definition bilinearly to deal with general elements of $\mathcal{L}$, satisfying the bilinearity property. We have

$$[L_n, L_n] = (n-n)L_{n+n} + \frac{c}{12}(n^3 - n)\delta_{n+n,0} = 0. \tag{4.51}$$

Note that the second term is zero since $\delta_{2n,0}$ vanishes unless $n = 0$, in which case $n^3 - n = 0$. Thus we satisfy (2) of Definition 4.2. Lastly, we will check that the Jacobi identity holds.

We have

$$[L_m,[L_n,L_r]]+[L_n,[L_r,L_m]]+[L_r,[L_m,L_n]] = \Big[L_m,(n-r)L_{n+r}+\frac{c}{12}(n^3-n)\delta_{n+r,0}\Big] \tag{4.52}$$

$$+\Big[L_n,(r-m)L_{r+m}+\frac{c}{12}(r^3-r)\delta_{r+m,0}\Big]+\Big[L_r,(m-n)L_{m+n}+\frac{c}{12}(m^3-m)\delta_{m+n,0}\Big] \tag{4.53}$$

$$=(n-r)[L_m,L_{n+r}]+(r-m)[L_n,L_{r+m}]+(m-n)[L_r,L_{m+n}] \tag{4.54}$$

$$=(n-r)\Big((m-(n+r))L_{m+(n+r)}+\frac{c}{12}(m^3-m)\delta_{m+(n+r),0}\Big) \tag{4.55}$$

$$+(r-m)\Big((n-(r+m))L_{n+(r+m)}+\frac{c}{12}(n^3-n)\delta_{n+(r+m),0}\Big) \tag{4.56}$$

$$+(m-n)\Big((r-(m+n))L_{r+(m+n)}+\frac{c}{12}(r^3-r)\delta_{r+(m+n),0}\Big) \tag{4.57}$$

$$=((r^2-n^2+mn-mr)+(m^2-r^2+nr-mn)+(n^2-m^2+mr-nr))L_{m+n+r} \tag{4.58}$$

$$+((n-r)(m^3-m)+(r-m)(n^3-n)+(m-n)(r^3-r))\frac{c}{12}\delta_{m+n+r,0} \tag{4.59}$$

$$=((m^3n-mn-m^3r+mr)+(n^3r-nr-mn^3+mn) \tag{4.60}$$

$$+(mr^3-mr-nr^3+nr))\frac{c}{12}\delta_{m+n+r,0} \tag{4.61}$$

$$=(m^3(n-r)+n^3(r-m)+r^3(m-n))\frac{c}{12}\delta_{m+n+r,0} \tag{4.62}$$

If $m+n+r\neq 0$, then the above expression evaluates to zero. Now consider the case

$m + n + r = 0$. Then $r = -(m + n)$ and we write the coefficient of the expression as

$$m^3(n - r) + n^3(r - m) + r^3(m - n) \tag{4.63}$$

$$= m^3(n + (m + n)) + n^3(-(m + n) - m) + -(m + n)^3(m - n) \tag{4.64}$$

$$= m^3(m + 2n) + n^3(-n - 2m) - (m + n)^3(m - n) \tag{4.65}$$

$$= (m^4 + 2m^3n) + (-n^4 - 2mn^3) - (m^3 + n^3 + 3mn(m + n))(m - n) \tag{4.66}$$

$$= (m^4 + 2m^3n) + (-n^4 - 2mn^3) - (m^4 - m^3n + mn^3 - n^4 + 3mn(m^2 - n^2)) \tag{4.67}$$

$$= m^4 + 2m^3n - n^4 - 2mn^3 - m^4 + m^3n - mn^3 + n^4 - 3m^3n + 3mn^3 = 0. \tag{4.68}$$

We conclude that the Jacobi identity is satisfied since the expression is zero in either case. Therefore $\mathcal{L}$ is a Lie algebra with the given bracket. □

We also define a particular class of modules of the Virasoro algebra called restricted modules.

Definition 4.7. Let W be a module of the Virasoro algebra. W is *restricted* if for every $w \in W$ we have $L(n)w = 0$ for n sufficiently large.

4.4 Construction of the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$

We construct the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$, denoted $U(\mathfrak{sl}_2(\mathbb{C})$, for completeness. We will define it as the quotient of the tensor algebra $T(\mathfrak{sl}_2(\mathbb{C}))$ by an equivalence relation $\sim$ defined in terms of the tensor product and the Lie bracket for $\mathfrak{sl}_2(\mathbb{C})$. We have the following relevant definitions.

Definition 4.8. The *tensor algebra* associated to $\mathfrak{sl}_2(\mathbb{C})$, denoted $T(\mathfrak{sl}_2(\mathbb{C}))$, is given by

$$\mathbb{C} \oplus \bigoplus_{n \geq 1} \mathfrak{sl}_2(\mathbb{C})^{\otimes n}. \tag{4.69}$$

Definition 4.9. We can lift the bracket $[\cdot,\cdot] : \mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$ to a bracket $[\cdot,\cdot] : T(\mathfrak{sl}_2(\mathbb{C})) \otimes T(\mathfrak{sl}_2(\mathbb{C})) \to T(\mathfrak{sl}_2(\mathbb{C}))$ in the following way. Define the bracket on $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$ as $a \otimes b - b \otimes a = [a, b]$. We then extend this definition recursively using the following rules to get the bracket on $T(\mathfrak{sl}_2(\mathbb{C})) \otimes T(\mathfrak{sl}_2(\mathbb{C})) \to T(\mathfrak{sl}_2(\mathbb{C}))$:

$$[a \otimes b, c] = a \otimes [b, c] + [a, c] \otimes b \tag{4.70}$$

$$[a, b \otimes c] = [a, b] \otimes c + b \otimes [a, c]. \tag{4.71}$$

Now define $\sim$ by $a \otimes b - b \otimes a = [a, b]$ for all $a, b \in T(\mathfrak{sl}_2(\mathbb{C}))$. Then we define the *universal enveloping algebra* of $\mathfrak{sl}_2(\mathbb{C})$ to be the quotient

$$U(\mathfrak{sl}_2(\mathbb{C})) \equiv T(\mathfrak{sl}_2(\mathbb{C}))/\sim . \tag{4.72}$$

4.5 Construction of the universal enveloping algebra $U(\hat{\mathfrak{sl}}_2(\mathbb{C}))$

The next step is to construct the universal enveloping algebra $U(\hat{\mathfrak{sl}}_2(\mathbb{C}))$ in preparation for defining $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)$. We can define it as the quotient of the tensor algebra $T(\hat{\mathfrak{sl}}_2(\mathbb{C}))$ by an equivalence relation $\sim$ defined in terms of the tensor product and the Lie bracket for $\hat{\mathfrak{sl}}_2(\mathbb{C})$. We have the following definitions.

Definition 4.10. The *tensor algebra* associated to $\hat{\mathfrak{sl}}_2(\mathbb{C})$, denoted $T(\hat{\mathfrak{sl}}_2(\mathbb{C}))$, is given by

$$T(\hat{\mathfrak{sl}}_2(\mathbb{C})) \equiv \mathbb{C} \oplus \bigoplus_{n\geq 1} \hat{\mathfrak{sl}}_2(\mathbb{C})^{\otimes n}. \tag{4.73}$$

Definition 4.11. We can lift the bracket $[\cdot,\cdot] : \hat{\mathfrak{sl}}_2(\mathbb{C}) \otimes \hat{\mathfrak{sl}}_2(\mathbb{C}) \to \hat{\mathfrak{sl}}_2(\mathbb{C})$ to a bracket $[\cdot,\cdot] : T(\hat{\mathfrak{sl}}_2(\mathbb{C})) \otimes T(\hat{\mathfrak{sl}}_2(\mathbb{C})) \to T(\hat{\mathfrak{sl}}_2(\mathbb{C}))$ in the following way. Define the bracket on $\hat{\mathfrak{sl}}_2(\mathbb{C}) \otimes \hat{\mathfrak{sl}}_2(\mathbb{C}) \to \hat{\mathfrak{sl}}_2(\mathbb{C})$ as $a \otimes b - b \otimes a = [a, b]$. We then extend this definition recursively using the following rules to get the bracket on $T(\hat{\mathfrak{sl}}_2(\mathbb{C})) \otimes T(\hat{\mathfrak{sl}}_2(\mathbb{C})) \to T(\hat{\mathfrak{sl}}_2(\mathbb{C}))$:

$$[a \otimes b, c] = a \otimes [b, c] + [a, c] \otimes b, \tag{4.74}$$

$$[a, b \otimes c] = [a, b] \otimes c + b \otimes [a, c]. \tag{4.75}$$

Now define $\sim$ by $a \otimes b - b \otimes a = [a, b]$ for all $a, b \in T(\hat{\mathfrak{sl}}_2(\mathbb{C}))$. Then we define the *universal enveloping algebra* of $\hat{\mathfrak{sl}}_2(\mathbb{C})$ to be the quotient

$$U(\hat{\mathfrak{sl}}_2(\mathbb{C})) \equiv T(\hat{\mathfrak{sl}}_2(\mathbb{C}))/\sim. \tag{4.76}$$

Chapter 5: General results of vertex algebra theory

In this chapter I follow the proofs given in [LL04] of some basic results in vertex algebra theory that will be necessary to construct the more complicated WZW model example of a vertex operator algebra.

5.1 Weak commutativity and the Jacobi identity

In this section I will prove some results related to weak commutativity and the Jacobi identity that are needed to show that $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0)$ is a vertex algebra.

Lemma 5.1. (Proposition 3.2.1 of [LL04]) (weak commutativity) For $u, v \in V$, there exists a nonnegative integer k such that

$$B_k(x_1, -x_2)[Y(u,x_1), Y(v,x_2)] = 0. \tag{5.1}$$

Proof. We wish to make the expression on the right-hand side of the Jacobi identity vanish by clearing a suitable formal pole. Let $k \geq 0$. Multiplying the Jacobi identity by x_0^k and then taking the formal residue Res_{x_0} gives

$$B_k(x_1, -x_2)Y(u,x_1)Y(v,x_2) - B_k(x_1,-x_2)Y(v,x_2)Y(u,x_1) \tag{5.2}$$

$$= \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) x_0^k Y(Y(u,x_0)v, x_2). \tag{5.3}$$

Then the result follows from choosing k such that $u_n v = 0$ for $n \geq k$. □

Lemma 5.2. (Proposition 3.2.7 of [LL04]) Let V be a vertex operator algebra and let $u, v, w \in V$ and $w' \in V'$ be arbitrary. We have:

(a) (rationality of products) The formal series

$$\langle w', Y(u, x_1)Y(v, x_2)w\rangle = \sum_{m,n\in\mathbb{Z}} \langle w', u_m v_n w\rangle x_1^{-m-1} x_2^{-n-1} \tag{5.4}$$

lies in the image of the map ι_{12}:

$$\langle w', Y(u, x_1)Y(v, x_2)w\rangle = \iota_{12} f(x_1, x_2), \tag{5.5}$$

where the uniquely determined element $f \in \mathbb{C}[x_1, x_2]_S$ is of the form

$$f(x_1, x_2) = \frac{g(x_1, x_2)}{(x_1 - x_2)^k x_1^l x_2^m} \tag{5.6}$$

for some $g \in \mathbb{C}[x_1, x_2]$ and $k, l, m \in \mathbb{Z}$, where k depends only on u and v; it is independent of w and w'.

(b) (commutativity) We also have

$$\langle w', Y(v, x_2)Y(u, x_1)w\rangle = \iota_{21} f(x_1, x_2). \tag{5.7}$$

Proof. Let k be a nonnegative integer such that the weak commutativity relation holds for u and v. Then

$$B_k(x_1, -x_2)\langle w', Y(u, x_1)Y(v, x_2)w\rangle = B_k(x_1, -x_2)\langle w', Y(v, x_2)Y(u, x_1)w\rangle. \tag{5.8}$$

The left-hand side involves only finitely many negative powers of x_2, by the truncation condition, and only finitely many positive powers of x_1, by

$$\text{wt } v_n = \text{wt } v - n - 1. \tag{5.9}$$

Similarly, the right-hand side involves only finitely many negative powers of x_1 and only finitely many positive powers of x_2. Therefore by the equality both sides have only finitely many positive and negative powers of both x_1 and x_2, so each side is a Laurent polynomial, that is, a formal Laurent series that is an element $h(x_1, x_2)$ of $\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]$. Then the rational function

$$f(x_1, x_2) = \frac{h(x_1, x_2)}{(x_1 - x_2)^k} \tag{5.10}$$

meets the conditions. □

Lemma 5.3. (Proposition 3.2.12 of [LL04]) (formal commutativity) Let V and W be vector spaces and suppose that we have a linear map $Y_W(\cdot, x) : V \to (\text{End } W)[[x, x^{-1}]]$. Let $u, v \in V$, $w \in W$, and $w^* \in W^*$. Assume that the following truncation conditions hold: $Y_W(u, x)w \in W((x))$ and $Y_W(v, x)w \in W((x))$. Assume also that weak commutativity holds for the pair (u, v) acting on the vector w, i.e. there exists $k \in \mathbb{N}$ depending only on u and v such that

$$B_k(x_1, -x_2)[Y_W(u, x_1), Y_W(v, x_2)]w = 0. \tag{5.11}$$

Then the formal series

$$\langle w^*, Y_W(u, x_1)Y_W(v, x_2)w\rangle \tag{5.12}$$

lies in the image of the map ι_{12}:

$$\langle w^*, Y_W(u, x_1)Y_W(v, x_2)w\rangle = \iota_{12} f(x_1, x_2), \tag{5.13}$$

where the uniquely determined element

$$f \in \mathbb{C}[[x_1, x_2]][x_1^{-1}, x_2^{-1}, (x_1 - x_2)^{-1}] \tag{5.14}$$

is of the form

$$f(x_1, x_2) = \frac{g(x_1, x_2)}{(x_1 - x_2)^k x_1^l x_2^m} \tag{5.15}$$

for some $g \in \mathbb{C}[[x_1, x_2]]$ and $k, l, m \in \mathbb{Z}$, where k depends only on u and v (and not on w or w^*). Moreover,

$$\langle w^*, Y_W(v, x_2)Y_W(u, x_1)w\rangle = \iota_{21}f(x_1, x_2). \tag{5.16}$$

Proof. The proof is similar to that of Lemma 5.2, albeit the formal series

$$B_k(x_1, -x_2)\langle w^*, Y_W(u, x_1)Y_W(v, x_2)w\rangle = B_k(x_1, -x_2)\langle w^*, Y_W(v, x_2)Y_W(u, x_1)w\rangle \tag{5.17}$$

lies in $\mathbb{C}((x_1, x_2))$ instead of $\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}]$. □

Lemma 5.4. (Proposition 3.3.1 of [LL04]) (weak associativity) For $u, w \in V$, there exists a nonnegative integer l (depending only on u and w) such that for every $v \in V$ we have

$$B_l(x_0, x_2)Y(Y(u, x_0)v, x_2)w = B_l(x_0, x_2)Y(u, x_0 + x_2)Y(v, x_2)w. \tag{5.18}$$

Proof. We use a similar approach to the proof of Lemma 5.1. Intuitively, we will make the second term on the left-hand side of the Jacobi identity vanish by clearing a suitable pole. By the formal delta function property

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right), \tag{5.19}$$

we have

$$x_1^{-1}\delta\left(\frac{x_0 + x_2}{x_1}\right)Y(u, x_1)Y(v, x_2) - x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)Y(Y(u, x_0)v, x_2) \tag{5.20}$$

$$= x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1). \tag{5.21}$$

We replace $Y(u,x_1)$ with $Y(u,x_0+x_2)$ in the first term on the left-hand side by

$$x_1^{-1}\delta\left(\frac{x_2-y}{x_1}\right)f(x_1,x_2,y)=x_1^{-1}\delta\left(\frac{x_2-y}{x_1}\right)f(x_2-y,x_2,y). \tag{5.22}$$

Now let $w\in V$ and let $l\geq 0$. Multiplying our (modified) Jacobi identity by x_1^l and taking the residue Res_{x_1} gives

$$B_l(x_0,x_2)(Y(u,x_0+x_2)Y(v,x_2)w-Y(Y(u,x_0)v,x_2)w) \tag{5.23}$$

$$=\mathrm{Res}_{x_1}x_1^l x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)w. \tag{5.24}$$

Then the result follows by choosing l such that $u_n w=0$ for $n\geq l$. □

Lemma 5.5. (Proposition 3.3.5 of [LL04]) Let V be a vertex operator algebra and let $u,v,w\in V$ and $w'\in V'$ be arbitrary. We have:

(a) (rationality of iterates) The formal series

$$\langle w',Y(Y(u,x_0)v,x_2)w\rangle=\sum_{m,n\in\mathbb{Z}}\langle w',(u_m v)_n w\rangle x_0^{-m-1}x_2^{-n-1} \tag{5.25}$$

lies in the image of the map ι_{20}:

$$\langle w',Y(Y(u,x_0)v,x_2)w\rangle=\iota_{20}p(x_0,x_2), \tag{5.26}$$

where the uniquely determined element $p\in\mathbb{C}[x_0,x_2]_S$ is of the form

$$p(x_0,x_2)=\frac{q(x_0,x_2)}{x_0^k(x_0+x_2)^l x_2^m} \tag{5.27}$$

for some $q \in \mathbb{C}[x_0, x_2]$ and $k, l, m \in \mathbb{Z}$, where l depends only on u and w; it is independent of v and w'.

(b) The series

$$\langle w', Y(u, x_0 + x_2)Y(v, x_2)w\rangle = \sum_{m,n\in\mathbb{Z}} \sum_{i\geq 0} \binom{-m-1}{i} \langle w', u_m v_n w\rangle x_0^{-m-1-i} x_2^{i-n-1} \quad (5.28)$$

lies in the image of ι_{02}:

$$\langle w', Y(u, x_0 + x_2)Y(v, x_2)w\rangle = \iota_{02} p(x_0, x_2). \quad (5.29)$$

Proof. Let l be a nonnegative integer such that the weak associativity relation

$$B_l(x_0, x_2)Y(Y(u, x_0)v, x_2)w = B_l(x_0, x_2)Y(u, x_0 + x_2)Y(v, x_2)w \quad (5.30)$$

holds. Then we have

$$B_l(x_0, x_2)\langle w', Y(u, x_0 + x_2)Y(v, x_2)w\rangle = B_l(x_0, x_2)\langle w', Y(Y(u, x_0)v, x_2)w\rangle. \quad (5.31)$$

The left-hand side involves only finitely many negative powers of x_2, by the truncation condition, and only finitely many positive powers of x_0, by

$$\text{wt } v_n = \text{wt } v - n - 1. \quad (5.32)$$

Similarly, the right-hand side involves only finitely many negative powers of x_0 and only finitely many positive powers of x_2. As a result, both sides have finitely many powers in both x_0 and x_2, and so the common series is a formal Laurent series of the form $r(x_0, x_2) \in$

$\mathbb{C}[x_0, x_0^{-1}, x_2, x_2^{-1}]$. We conclude that the rational function

$$p(x_0, x_2) = \frac{r(x_0, x_2)}{(x_0 + x_2)^l} \tag{5.33}$$

satisfies the conditions. □

Lemma 5.6. (Proposition 3.3.8 of [LL04]) (associativity) We have

$$\iota_{12}^{-1}\langle w', Y(u, x_1)Y(v, x_2)w\rangle = (\iota_{20}^{-1}\langle w', Y(Y(u, x_0)v, x_2)w\rangle)|_{x_0=x_1-x_2}. \tag{5.34}$$

Proof. By Lemma 5.2, we have $f(x_1, x_2) = p(x_1 - x_2, x_2)$. The result immediately follows from this. □

Lemma 5.7. (Proposition 3.3.17 of [LL04]) Let V and W be vector spaces equipped with a linear map

$$Y(\cdot, x) : V \to (\text{End } V)[[x, x^{-1}]] \tag{5.35}$$

and also a linear map

$$Y_W(\cdot, x) : V \to (\text{End } W)[[x, x^{-1}]]. \tag{5.36}$$

Let $u, v \in V$, $w \in W$, and $w^* \in W^*$. Assume that the truncation conditions $Y(u, x)v \in V((x))$ and $Y_W(v, x)w \in W((x))$ hold and assume also that weak associativity holds for (u, v, w), i.e. there exists a nonnegative integer l that depends only on u and w so that

$$B_l(x_0, x_2)Y_W(Y(u, x_0)v, x_2)w = B_l(x_0, x_2)Y_W(u, x_0 + x_2)Y_W(v, x_2)w. \tag{5.37}$$

Then the formal series

$$\langle w^*, Y_W(Y(u, x_0)v, x_2)w\rangle \tag{5.38}$$

lies in the image of the map ι_{20}:

$$\langle w^*, Y_W(Y(u, x_0)v, x_2)w\rangle = \iota_{20}p(x_0, x_2), \tag{5.39}$$

where the uniquely determined element

$$p \in \mathbb{C}[[x_0, x_2]][x_0^{-1}, x_2^{-1}, (x_0 + x_2)^{-1}] \tag{5.40}$$

is of the form

$$p(x_0, x_2) = \frac{q(x_0, x_2)}{x_0^k (x_0 + x_2)^l x_2^m} \tag{5.41}$$

for some $q \in \mathbb{C}[[x_0, x_2]]$ and $k, l, m \in \mathbb{Z}$, where l depends only on u and w (and not on v or w^*). Moreover,

$$\langle w^*, Y_W(u, x_0 + x_2)Y_W(v, x_2)w\rangle = \iota_{02}p(x_0, x_2). \tag{5.42}$$

Proof. The proof is analogous to the proof of Lemma 5.5. □

Lemma 5.8. (Proposition 3.3.19 of [LL04]) (formal associativity) Under the hypotheses of Lemma 5.7, with $u, v \in V$, $w \in W$ and $w^* \in W^*$, the formal series

$$\langle w^*, Y_W(u, x_1)Y_W(v, x_2)w\rangle \tag{5.43}$$

lies in the image of the map ι_{12}:

$$\langle w^*, Y_W(u, x_1)Y_W(v, x_2)w\rangle = \iota_{12}f(x_1, x_2), \tag{5.44}$$

where the uniquely determined element

$$f \in \mathbb{C}[[x_1, x_2]][x_1^{-1}, x_2^{-1}, (x_1 - x_2)^{-1}] \tag{5.45}$$

is of the form

$$f(x_1, x_2) = \frac{g(x_1, x_2)}{(x_1 - x_2)^k x_1^l x_2^m} \tag{5.46}$$

for some $g \in \mathbb{C}[[x_1, x_2]]$ and $k, l, m \in \mathbb{Z}$, where l depends only on u and w (and not on v or w^*). Moreover,

$$f(x_1, x_2) = p(x_1 - x_2, x_2), \tag{5.47}$$

or equivalently,

$$\iota_{12}^{-1}\langle w^*, Y_W(u, x_1)Y_W(v, x_2)w\rangle = (\iota_{20}^{-1}\langle w^*, Y_W(Y(u, x_0)v, x_2)w\rangle)|_{x_0 = x_1 - x_2} \tag{5.48}$$

(as elements of $\mathbb{C}[[x_1, x_2]][x_1^{-1}, x_2^{-1}, (x_1 - x_2)^{-1}]$).

Proof. The proof is similar to the proof of Lemma 5.2. We have a well-defined canonical map

$$\mathbb{C}[[x_1, x_2]][x_1^{-1}, x_2^{-1}, (x_1 - x_2)^{-1}] \to \mathbb{C}[[x_0, x_2]][x_0^{-1}, x_2^{-1}, (x_0 + x_2)^{-1}] \tag{5.49}$$

$$s(x_1, x_2) \mapsto s(x_1, x_2)|_{x_1 = x_0 + x_2} = s(x_0 + x_2, x_2), \tag{5.50}$$

and the reverse map

$$t(x_0, x_2) \mapsto t(x_0, x_2)|_{x_0 = x_1 - x_2} = t(x_1 - x_2, x_2) \tag{5.51}$$

is the inverse map. Moreover, for such $t(x_0, x_2)$ we have

$$\iota_{12} t(x_1 - x_2, x_2) = (\iota_{02} t(x_0, x_2))|_{x_0 = x_1 - x_2}. \tag{5.52}$$

The rest is clear from the proof of Lemma 5.2. □

Lemma 5.9. (Proposition 3.4.1 of [LL04]) The Jacobi identity for a vertex operator algebra follows from the rationsality of products and iterates, and commutativity and associativity.

In particular, in the definition of the notion of vertex operator algebra, the Jacobi identity can be replaced by these properties.

Proof. For a formal Laurent polynomial $f(x_0, x_1, x_2)$ (that is, a formal Laurent series of the form

$$f(x_0, x_1, x_2) = \frac{g(x_0, x_1, x_2)}{x_0^r x_1^s x_2^t} \tag{5.53}$$

where g is a polynomial and $r, s, t \in \mathbb{Z}$), take $g(x_0, x_1)/x_0^k x_1^l x_2^m$ in the notation of Lemma 5.2. Then by Lemma 5.2,

$$\iota_{12}(f|_{x_0=x_1-x_2}) = \langle w', Y(u, x_1)Y(v, x_2)w\rangle \tag{5.54}$$

and

$$\iota_{21}(f|_{x_0=x_1-x_2}) = \langle w', Y(v, x_2)Y(u, x_1)w\rangle, \tag{5.55}$$

and by Lemmas 5.5(a) and 5.6, we have

$$\iota_{20}(f|_{x_1=x_0+x_2}) = \iota_{20}(f(x_1 - x_2, x_1, x_2)|_{x_1=x_0+x_2}) \tag{5.56}$$

$$= \iota_{20}((\iota_{12}^{-1}\langle w', Y(u, x_1)Y(v, x_2)w\rangle)|_{x_1=x_0+x_2}) \tag{5.57}$$

$$= \langle w', Y(Y(u, x_0)v, x_2)w\rangle. \tag{5.58}$$

The Jacobi identity then follows. □

Lemma 5.10. (Proposition 3.4.3 of [LL04]) Let V and W be vector spaces and suppose we have a linear map

$$Y_W(\cdot, x) : V \to (\text{End } W)[[x, x^{-1}]] \tag{5.59}$$

$$v \mapsto Y_W(v, x). \tag{5.60}$$

Assume that $Y_W(u,x)w, Y_W(v,x)w \in W((x))$ for any $u, v \in V$ and $w \in W$. Also assume that weak commutativity holds for the pair (u,v) acting on w, i.e., there exists $k \in \mathbb{N}$ such that

$$B_k(x_1, -x_2)[Y_W(u,x_1), Y_W(v,x_2)]w = 0. \tag{5.61}$$

Further assume that $Y(u,x)v \in V((x))$ for all $u, v \in V$. Then the Jacobi identity

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u,x_1)Y_W(v,x_2)w - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(v,x_2)Y_W(u,x_1)w \tag{5.62}$$

$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y(u,x_0)v, x_2)w \tag{5.63}$$

is equivalent to weak commutativity for the pair (u,v) acting on w together with weak associativity for the triple (u,v,w). In particular, in the definition of vertex algebra, the Jacobi identity can be replaced by weak commutativity and weak associativity.

Proof. The Jacobi identity for (u,v,w) implies weak commutativity and weak associativity for (u,v,w) by the proofs of Lemmas 5.1 and 5.4. Conversely, using formal commutativity and formal associativity we can see that due to Theorem 1.37, the proof of Lemma 5.9 still works. □

Theorem 5.11. (Theorem 3.5.1 of [LL04]) The Jacobi identity for a vertex algebra follows from weak commutativity in the presence of other axioms together with the $\mathcal{D}$-bracket-derivative formula

$$[\mathcal{D}, Y(v,x)] = \frac{d}{dx}Y(v,x) \tag{5.64}$$

for $v \in V$. In particular, in the definition of the notion of vertex algebra, the Jacobi identity can be replaced by these properties.

Proof. We will show that skew symmetry and then weak associativity holds, and the Jacobi identity subsequently follows from Lemma 5.10.

For skew symmetry we will first show

$$Y(v,x)\mathbf{1} = e^{x\mathcal{D}}v \tag{5.65}$$

for all $v \in V$, which is a special case, and then proceed to the general case. By the vacuum property we have $\mathcal{D}(\mathbf{1}) = \mathbf{1}_{-2}\mathbf{1} = 0$, where $\mathbf{1}_n$ for $n \in \mathbb{Z}$ is defined by

$$Y(\mathbf{1},x) = \sum_{n\in\mathbb{Z}} \mathbf{1}_n x^{-n-1}.$$

The $\mathcal{D}$-bracket-derivative formula implies

$$e^{x_0\mathcal{D}}Y(v,x)e^{-x_0\mathcal{D}} = Y(v,x+x_0) \tag{5.66}$$

for all $v \in V$, which in turn implies

$$Y(v,x_0+x)\mathbf{1} = e^{x\mathcal{D}}Y(v,x_0)e^{-x\mathcal{D}}\mathbf{1} = e^{x\mathcal{D}}Y(v,x_0)\mathbf{1}. \tag{5.67}$$

Due to the creation property we can set $x_0 = 0$ and obtain $Y(v,x)\mathbf{1} = e^{x\mathcal{D}}v$.

Now we will prove skew symmetry in the general case. Let $u, v \in V$ and let $k \in \mathbb{N}$ such that $x^k Y(v,x)u$ involves only nonnegative powers of x and the weak commutativity relation

$$B_k(x_1,-x_2)[Y(u,x_1),Y(v,x_2)] = 0 \tag{5.68}$$

holds. By

$$Y(v,x)\mathbf{1} = e^{x\mathcal{D}}v \tag{5.69}$$

and

$$e^{x_0\mathcal{D}}Y(v,x)e^{-x_0\mathcal{D}} = Y(v,x+x_0) \tag{5.70}$$

we have

$$B_k(x_1,-x_2)Y(u,x_1)Y(v,x_2)\mathbf{1} = B_k(x_1,-x_2)Y(v,x_2)Y(u,x_1)\mathbf{1} \tag{5.71}$$

$$= B_k(x_1,-x_2)Y(v,x_2)e^{x_1\mathcal{D}}u \tag{5.72}$$

$$= B_k(x_1,-x_2)e^{x_1\mathcal{D}}Y(v,x_2-x_1)u. \tag{5.73}$$

In the last expression of the above derivation we can set $x_2 = 0$ since $B_k(x_1,-x_2)Y(v,x_2-x_1)u$ involves only nonnegative powers of $x_2 - x_1$. Once we do this and use the creation property we have

$$x_1^k Y(u,x_1)v = x_1^k e^{x_1\mathcal{D}}Y(v,-x_1)u. \tag{5.74}$$

Multiplying both sides by x_1^{-k} results in the skew symmetry relation.

Now we show that weak associativity holds as well. For any $u,v,w \in V$, let $k \in \mathbb{N}$ be such that weak commutativity holds for u and w. Then by applying skew symmetry to the pairs (v,w) and $(Y(u,x_0)v,w)$, and the formula

$$e^{x_0\mathcal{D}}Y(v,x)e^{-x_0\mathcal{D}} = Y(v,x+x_0), \tag{5.75}$$

we obtain weak associativity via

$$B_k(x_0,x_2)Y(u,x_0+x_2)Y(v,x_2)w = B_k(x_0,x_2)Y(u,x_0+x_2)e^{x_2\mathcal{D}}Y(w,-x_2)v \tag{5.76}$$

$$= e^{x_2\mathcal{D}}B_k(x_0,x_2)Y(u,x_0)Y(w,-x_2)v \tag{5.77}$$

$$= e^{x_2\mathcal{D}}B_k(x_0,x_2)Y(w,-x_2)Y(u,x_0)v \tag{5.78}$$

$$= B_k(x_0,x_2)Y(Y(u,x_0)v,x_2)w. \tag{5.79}$$

□

Theorem 5.12. (Theorem 3.6.3 of [LL04]) Let $(V, Y, \mathbf{1})$ satisfy all the axioms in the definition of a vertex algebra except for the Jacobi identity, and in addition has the skew symmetry property

$$Y(u,x)v = e^{x\mathcal{D}}Y(v,-x)u \tag{5.80}$$

for $u, v \in V$. Let W be a vector space and let $Y_W(\cdot, x)$ be a linear map from V to $(\text{End } W)[[x, x^{-1}]]$ such that $Y_W(\mathbf{1}, x) = \text{id}_W$ and $Y_W(v, x)w \in W((x))$ for $v \in V$ and $w \in W$. Assume too that weak associativity holds for any $u, v \in V$ and $w \in W$, in the sense that there exists $l \in \mathbb{N}$ depending on u, v, and w such that

$$B_l(x_0, x_2)Y_W(Y(u, x_0)v, x_2)w = B_l(x_0, x_2)Y_W(u, x_0 + x_2)Y_W(v, x_2)w. \tag{5.81}$$

Then the Jacobi identity

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u, x_1)Y_W(v, x_2)w - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(v, x_2)Y_W(u, x_1)w \tag{5.82}$$

$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y(u, x_0)v, x_2)w \tag{5.83}$$

holds for any $u, v \in V$ and $w \in W$.

Proof. First we prove the $\mathcal{D}$-derivative formula

$$Y_W(\mathcal{D}v, x) = \frac{d}{dx}Y_W(v, x). \tag{5.84}$$

For $u \in V$ and $w \in W$, by the weak associativity assumption there exists $l \in \mathbb{N}$ such that

$$B_l(x_0, x_2)Y_W(Y(u, x_0)\mathbf{1}, x_2)w = B_l(x_0, x_2)Y_W(u, x_0 + x_2)Y_W(\mathbf{1}, x_2)w. \tag{5.85}$$

Using the vacuum property for $\mathbf{1}$ acting on W and the formula $Y(u, x)\mathbf{1} = e^{x\mathcal{D}}u$, which follows from the vacuum property for V and skew symmetry, we obtain

$$B_l(x_0, x_2)Y_W(e^{x_0\mathcal{D}}u, x_2)w = B_l(x_0, x_2)Y_W(u, x_0 + x_2)w. \tag{5.86}$$

From the assumed truncation condition, we choose l larger if needed so that $x^l Y_W(u, x)w \in W[[x]]$, and we have

$$B_l(x_0, x_2)Y_W(e^{x_0\mathcal{D}}u, x_2)w = B_l(x_0, x_2)Y_W(u, x_2 + x_0)w, \tag{5.87}$$

since we can now change $x_0 + x_2$ to $x_2 + x_0$ on the right-hand side. Because all the factors involve only nonnegative powers of x_0, we can multiply by $B_{-l}(x_2, x_0)$ to get

$$Y_W(e^{x_0\mathcal{D}}u, x_2)w = Y_W(u, x_2 + x_0)w, \tag{5.88}$$

which is the global form of our desired $\mathcal{D}$-derivative formula.

Now we will prove the Jacobi identity. By Lemmas 5.7 and 5.8, formal associativity holds for Y_W. Then by the proof of Lemma 5.10, it is enough to show that for $u, v \in V$, $w \in W$, and $w^* \in W^*$, there exists

$$f(x_1, x_2) \in \mathbb{C}[[x_1, x_2]][x_1^{-1}, x_2^{-1}, (x_1 - x_2)^{-1}] \tag{5.89}$$

such that

$$\langle w^*, Y_W(u, x_1)Y_W(v, x_2)w\rangle = \iota_{12} f(x_1, x_2), \tag{5.90}$$

$$\langle w^*, Y_W(v, x_2)Y_W(u, x_1)w\rangle = \iota_{21} f(x_1, x_2). \tag{5.91}$$

By Lemmas 5.7 and 5.8 there exists a uniquely determined element

$$f(x_1, x_2) \in \mathbb{C}[[x_1, x_2]][x_1^{-1}, x_2^{-1}, (x_1 - x_2)^{-1}]$$

such that

$$\langle w^*, Y_W(u,x_1)Y_W(v,x_2)w\rangle = \iota_{12} f(x_1,x_2), \tag{5.92}$$

$$\langle w^*, Y_W(Y(u,x_0)v,x_2)w\rangle = \iota_{20} f(x_0+x_2,x_2). \tag{5.93}$$

Similarly, reversing the roles of u and v, x_1 and x_2, and x_0 and $-x_0$ yields a unique element

$$f'(x_1,x_2) \in \mathbb{C}[[x_1,x_2]][x_1^{-1},x_2^{-1},(x_1-x_2)^{-1}] \tag{5.94}$$

such that

$$\langle w^*, Y_W(v,x_2)Y_W(u,x_1)w\rangle = \iota_{21} f'(x_1,x_2), \tag{5.95}$$

$$\langle w^*, Y_W(Y(v,-x_0)u,x_1)w\rangle = \iota_{10} f'(x_1,x_1-x_0). \tag{5.96}$$

It now suffices to show that $f = f'$.

By the assumed skew symmetry and the $\mathcal{D}$-derivative formula for Y_W we have

$$\langle w^*, Y_W(Y(u,x_0)v,x_2)w\rangle = \langle w^*, Y_W(e^{x_0\mathcal{D}}Y(v,-x_0)u,x_2)w\rangle \tag{5.97}$$

$$= \langle w^*, Y_W(Y(v,-x_0)u,x_2+x_0)w\rangle \tag{5.98}$$

$$= (\iota_{10} f'(x_1,x_1-x_0))|_{x_1=x_2+x_0} \tag{5.99}$$

$$= \iota_{20} f'(x_0+x_2,x_2). \tag{5.100}$$

By

$$\langle w^*, Y_W(Y(u,x_0)v,x_2)w\rangle = \iota_{20} f(x_0+x_2,x_2) \tag{5.101}$$

it then follows that $f = f'$. □

5.2 Generating vertex subalgebras by subsets

We will show that a subset of a vertex algebra generates a vertex subalgebra.

Theorem 5.13. (Proposition 3.9.3 of [LL04]) For a subset S of the vertex algebra V, we obtain a vertex subalgebra $\langle S\rangle$ generated by S and given by

$$\langle S\rangle = \mathrm{span}\{u^{(1)}_{n_1}\cdots u^{(r)}_{n_r}\mathbf{1} \,|\, r\in\mathbb{N}, u^{(1)},\ldots,u^{(r)}\in S, n_1,\ldots,n_r\in\mathbb{Z}\}. \tag{5.102}$$

Proof. Let U be the subspace given by

$$U = \mathrm{span}\{u^{(1)}_{n_1}\cdots u^{(r)}_{n_r}\mathbf{1} \,|\, r\in\mathbb{N}, u^{(1)},\ldots,u^{(r)}\in S, n_1,\ldots,n_r\in\mathbb{Z}\}. \tag{5.103}$$

Clearly $S\subset U$ and any vertex subalgebra that contains S contains U, so it suffices to prove that U is a vertex subalgebra of V. Since $\mathbf{1}\in U$, we need to show that $Y(a,x)b\in U((x))$ for $a,b\in U$. Set

$$K = \{a\in U \,|\, Y(a,x)U\subset U((x))\}. \tag{5.104}$$

We need to show that $K=U$. By the definitions we have given, $\{\mathbf{1}\}\cup S\subset K$. Let $a\in K$ and $u\in S$. Then it follows from the iterate formula

$$Y(Y(u,x_0)a,x_2) = \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(a,x_2) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(a,x_2)Y(u,x_1)\right) \tag{5.105}$$

(Lemma 2.11) and the assumption on a that

$$Y(Y(u,x_0)a,x_2)U\subset U[[x_0,x_0^{-1},x_2,x_2^{-1}]], \tag{5.106}$$

so that $Y(u,x_0)a$, which lies in $U((x_0))$, also lies in $K((x_0))$. By induction, $U\subset K$ and so $K=U$ as desired. $\square$

5.3 The Jacobi identity from weak associativity for modules

Lemma 5.14. (Proposition 4.2.1 of [LL04]) (weak commutativity) Let (W, Y_W) be a module of the vertex algebra V. Then for $u, v \in V$ there exists a nonnegative integer k such that

$$B_k(x_1, -x_2)[Y_W(u, x_1), Y_W(v, x_2)] = 0. \tag{5.107}$$

Proof. The proof is analogous to that of Lemma 5.1. □

Theorem 5.15. (Theorem 4.4.5 of [LL04]) Let V be a vertex algebra, W a vector space, and Y_W a linear map $V \to (\text{End } W)[[x, x^{-1}]]$ such that $Y_W(\mathbf{1}, x) = \text{id}_W$, $Y_W(v, x)w \in W((x))$ for $v \in V$, $w \in W$, and weak associativity holds for Y_W. Then Y_W satisfies the Jacobi identity and (W, Y_W) is a V-module. In particular, in the definition of a V-module, the Jacobi identity can be replaced by weak associativity for Y_W.

Proof. The result follows immediately from Theorem 5.12. □

Corollary 5.16. (Corollary 4.4.7 of [LL04]) Let V be a vertex algebra, W a vector space, and Y_W a linear map $V \to (\text{End } W)[[x, x^{-1}]]$ such that $Y_W(\mathbf{1}, x) = \text{id}_W$, $Y_W(v, x)w \in W((x))$ for $v \in V$, $w \in W$, and such that the iterate formula

$$Y_W(Y(u, x_0)v, x_2) \tag{5.108}$$

$$= \text{Res}_{x_1} \left(x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u, x_1)Y_W(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(v, x_2)Y_W(u, x_1) \right) \tag{5.109}$$

holds for $u, v \in V$. Then Y_W satisfies the Jacobi identity and (W, Y_W) is a V-module. In particular, in the definition of V-module we can replace the Jacobi identity with the iterate formula.

Proof. Let $w \in W$ and l be a nonnegative integer such that $x^l Y_W(u, x)w \in W[[x]]$. If we apply the iterate formula to w and multiply the resulting formula by $B_l(x_0, x_2)$, we have the weak associativity relation

$$B_l(x_0, x_2)Y_W(Y(u, x_0)v, x_2)w = B_l(x_0, x_2)Y_W(u, x_0 + x_2)Y_W(v, x_2)w. \tag{5.110}$$

The rest follows by Theorem 5.15. □

5.4 More results on modules

Lemma 5.17. (Proposition 4.5.7 of [LL04]) Let W be a module of a vertex algebra V. Let $u, v \in V$, $p, q \in \mathbb{Z}$, and $w \in W$. Then $u_p v_q w$ is a linear combination of elements of the form $t_r w$ where $t \in V$ and $r \in \mathbb{Z}$, and t can be chosen to have the form $u_s v$ for some $s \in \mathbb{Z}$. In particular, let l be a nonnegative integer such that $u_n w = 0$ for $n \geq l$. Let m be a nonnegative integer such that $v_n w = 0$ for $n > m + q$. Then we have

$$u_p v_q w = \sum_{i=0}^{m} \sum_{j=0}^{l} \binom{p-l}{i} \binom{l}{j} (u_{p-l-i+j} v)_{q+l+i-j} w. \tag{5.111}$$

Proof. Note that $u_n w = 0$ for $n \geq l$. This implies

$$B_l(x_0, x_2)Y(u, x_0 + x_2)Y(v, x_2)w = B_l(x_0, x_2)Y(Y(u, x_0)v, x_2)w. \tag{5.112}$$

Also note that

$$u_pv_qw = \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^p x_2^q Y(u,x_1)Y(v,x_2)w \tag{5.113}$$

$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)x_1^p x_2^q Y(u,x_1)Y(v,x_2)w \tag{5.114}$$

$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{-1}\delta\left(\frac{x_0+x_2}{x_1}\right)x_1^p x_2^q Y(u,x_1)Y(v,x_2)w \tag{5.115}$$

$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^{-1}\delta\left(\frac{x_0+x_2}{x_1}\right)B_p(x_0,x_2)x_2^q Y(u,x_0+x_2)Y(v,x_2)w \tag{5.116}$$

$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}B_p(x_0,x_2)x_2^q Y(u,x_0+x_2)Y(v,x_2)w. \tag{5.117}$$

By these two results we now have

$$u_pv_qw = \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}B_{p-l}(x_0,x_2)x_2^q[B_l(x_0,x_2)Y(u,x_0+x_2)Y(v,x_2)w] \tag{5.118}$$

$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}B_{p-l}(x_0,x_2)x_2^q[B_l(x_0,x_2)Y(Y(u,x_0)v,x_2)w]. \tag{5.119}$$

Thus u_pv_qw is a linear combination of elements of the form $(u_sv)_rw$.

Now we will derive the formula. Note that if we set

$$p(x_0,x_2) = B_{p-l}(x_0,x_2) = \sum_{i=0}^{m}\binom{p-l}{i}x_0^{p-l-i}x_2^i, \tag{5.120}$$

we obtain

$$u_pv_qw = \mathrm{Res}_{x_0}\mathrm{Res}_{x_2}p(x_0,x_2)x_2^qB_l(x_0,x_2)Y(Y(u,x_0)v,x_2)w, \tag{5.121}$$

and the formula (5.111) follows immediately. □

Lemma 5.18. (Proposition 4.5.6 of [LL04]) Let W be a module of a vertex algebra V. Let $T \subset W$. Then we have

$$\langle T \rangle = \text{span}_{\mathbb{C}}\{v_n w | v \in V, n \in \mathbb{Z}, w \in T\}, \tag{5.122}$$

where $\langle T \rangle$ is the smallest submodule containing T.

Proof. This follows immediately from Lemma 5.17. □

We now define an *ideal* of a vertex algebra.

Definition 5.19. An *ideal* of a vertex algebra V is a subspace I such that for every $v \in V$ and $w \in I$ we have $Y(v, x)w \in I((x))$ and $Y(w, x)v \in I((x))$.

To continue the parallels to commutative algebra, we also have the notion of an *annihilator* of an ideal.

Definition 5.20. Let $T \subset W$ with W a module of a vertex algebra V. Define the *annihilator* $\mathcal{I}_V(T)$ of T in V to be

$$\mathcal{I}_V(T) \equiv \{v \in V \,|\, Y(v, x)w = 0 \text{ for } w \in T\}. \tag{5.123}$$

As expected, the annihilator is an ideal.

Theorem 5.21. (Proposition 4.5.11 of [LL04]) Given a subset $T \subset W$ where W is a module of a vertex algebra V, the annihilator $\mathcal{I}_V(T)$ is an ideal of V, and we have

$$\mathcal{I}_V(T) = \mathcal{I}_V(\langle T \rangle). \tag{5.124}$$

Proof. By linearity of $Y(\cdot, x)$, $\mathcal{I}_V(T)$ is a subspace of V. We also have $\mathcal{D}\mathcal{I}_V(T) \subset \mathcal{I}_V(T)$ since for $v \in \mathcal{I}_V(T)$ and $w \in T$, we have

$$Y(\mathcal{D}v, x)w = \frac{d}{dx}Y(v, x)w = 0. \tag{5.125}$$

Now let $u \in V$ and $v \in \mathcal{I}_V(T)$. For $w \in T$, let l be a nonnegative integer such that weak associativity holds. Then by

$$B_l(x_0, x_2)Y(Y(u, x_0)v, x_2)w = B_l(x_0, x_2)Y(u, x_0 + x_2)Y(v, x_2)w = 0, \tag{5.126}$$

we obtain

$$Y(Y(u, x_0)v, x_2)w = 0, \tag{5.127}$$

in light of the fact that $Y(Y(u, x_0)v, x_2)w$ is truncated below in powers of x_0. It then follows that $Y(u, x)v \in \mathcal{I}_V(T)((x))$. By skew symmetry and $\mathcal{D}\mathcal{I}_V(T) \subset \mathcal{I}_V(T)$, this condition we have just shown is equivalent to $Y(v, x)u \in \mathcal{I}_V(T)((x))$, and so $\mathcal{I}_V(T)$ is an ideal.

Now we show that $\mathcal{I}_V(T) = \mathcal{I}_V(\langle T\rangle)$. The inclusion $\mathcal{I}_V(\langle T\rangle) \subset \mathcal{I}_V(T)$ is obvious. Let $v \in \mathcal{I}_V(T)$. Given $u \in V$ and $w \in T$, let k be a nonnegative integer so that weak commutativity holds. Then we have

$$B_k(x_1, -x_2)Y(v, x_1)Y(u, x_2)w = B_k(x_1, -x_2)Y(u, x_2)Y(v, x_1)w = 0, \tag{5.128}$$

showing that $Y(v, x_1)Y(u, x_2) = 0$. Then by Lemma 5.18, $v \in \mathcal{I}_V(\langle T\rangle)$ which shows the other inclusion. □

We also have the notion of an annihilator of a subset of V in a module W.

Definition 5.22. Let W be a V-module. Let $S \subset V$. Define

$$\mathcal{M}_W(S) \equiv \{w \in W | Y(v, x)w = 0 \text{ for } v \in S\} \tag{5.129}$$

to be the *annihilator* of S in W.

As expected, this is a submodule of W.

Theorem 5.23. (Proposition 4.5.14 of [LL04]) Given $S \subset V$, the annihilator $\mathcal{M}_W(S)$ is a submodule of W. Moreover,

$$\mathcal{M}_W(S) = \mathcal{M}_W((S)), \tag{5.130}$$

where as in commutative algebra (S) denotes the ideal of V generated by S.

Proof. It is obvious that $\mathcal{M}_W(S)$ is a subspace of W by linearity of $Y(\cdot, x)$. Now let $v \in V$ and $w \in \mathcal{M}_W(S)$. For $u \in S$, let k be a nonnegative integer such that weak commutativity holds. Then we have

$$B_k(x_1, -x_2)Y(u, x_1)Y(v, x_2)w = B_k(x_1, -x_2)Y(v, x_2)Y(u, x_1)w = 0, \tag{5.131}$$

which implies $Y(u, x_1)Y(v, x_2)w = 0$. It follows that $Y(v, x)w \in \mathcal{M}_W(S)((x))$, and as a result $\mathcal{M}_W(S)$ is a submodule of W.

As the inclusion $\mathcal{M}_W((S)) \subset \mathcal{M}_W(S)$ is obvious, we only need to show that $\mathcal{M}_W(S) \subset \mathcal{M}_W((S))$. Let $w \in \mathcal{M}_W(S)$. By Theorem 5.21, $\mathcal{I}_V(\{w\})$ is an ideal that includes S and therefore includes (S). Thus, $w \in \mathcal{M}_W((S))$, showing the inclusion $\mathcal{M}_W(S) \subset \mathcal{M}_W((S))$ holds. □

Now we introduce *vacuum-like vectors*, which are module elements that share propertied with the vacuum vector in the definition of a vertex algebra.

Definition 5.24. Let W be a module of a vertex algebra V. A *vacuum-like vector* $w \in W$ satisfies $v_n w = 0$ for all $v \in V$ and $n \geq 0$, that is, $Y(v, x)w$ has only nonnegative powers of x.

Before we present needed results on vacuum-like vectors, we will need the notion of a V-module with derivation.

Definition 5.25. Let (W, Y_W) be a V-module. We call (W, Y_W, d) a V-module with derivation, or simply a V-module, if $d \in \operatorname{End} W$ and for every $v \in V$ we have

$$[d, Y_W(v, x)] = Y_W(\mathcal{D}v, x) = \frac{d}{dx}Y_W(v, x). \tag{5.132}$$

Theorem 5.26. (Proposition 4.7.4 of [LL04]) Let (W, Y_W, d) be a V-module and let $w \in W$

with $dw = 0$. Then for every $v \in V$ we have

$$Y_W(v, x)w = e^{xd}v_{-1}w, \tag{5.133}$$

and w is a vacuum-like vector.

Proof. Let $v \in V$. If $Y_W(v, x)w = 0$, then the result is obvious. Suppose now that $Y_W(v, x)w \neq 0$. Then there exists an integer k so that

$$v_k w \neq 0, v_n w = 0 \tag{5.134}$$

for $n > k$. By

$$[d, Y_W(v, x)] = \frac{d}{dx}Y_W(v, x), \tag{5.135}$$

we obtain

$$[d, v_n] = -nv_{n-1} \tag{5.136}$$

for $n \in \mathbb{Z}$. It follows that $-(k+1)v_k w = [d, v_{k+1}]w = 0$. We know that $k = -1$ since otherwise $v_k w = 0$, a contradiction. It follows that $v_n w = 0$ for $n \geq 0$, showing that w is a vacuum-like vector.

Also, by $dw = 0$ and the d-bracket formula we obtain

$$e^{xd}Y_W(v, x_1)w = e^{xd}Y_W(v, x_1)e^{-xd}w = Y_W(v, x_1 + x)w. \tag{5.137}$$

Since $Y_W(v, x_1)$ involves only nonnegative powers of x_1 by w being a vacuum-like vector, we can set $x_1 = 0$ and get the formula. □

Theorem 5.27. (Proposition 4.7.7 of [LL04]) Let (W, Y_W) be a V-module. Let $w \in W$ be a vacuum-like vector. The linear map

$$f : V \to W \tag{5.138}$$

$$v \mapsto v_{-1}w \tag{5.139}$$

is a V-homomorphism, uniquely determined by $\mathbf{1} \mapsto w$.

Proof. Let $u, v \in V$. We have

$$Y(Y(u, x_0)v, x_2)w = Y(u, x_0 + x_2)Y(v, x_2)w. \tag{5.140}$$

Then

$$f(Y(u, x_0)v) = \mathrm{Res}_{x_2} x_2^{-1} Y(Y(u, x_0)v, x_2)w \tag{5.141}$$

$$= \mathrm{Res}_{x_2} x_2^{-1} Y(u, x_0 + x_2)Y(v, x_2)w \tag{5.142}$$

$$= \lim_{x_2 \to 0} Y(u, x_0 + x_2)Y(v, x_2)w \tag{5.143}$$

$$= Y(u, x_0)v_{-1}w = Y(u, x_0)f(v). \tag{5.144}$$

By definition we conclude that f is a V-module homomorphism. □

Theorem 5.28. (Proposition 4.7.9 of [LL04]) Let (W, Y_W, d) be a faithful V-module. Let $w \in W$ be such that w generates W as a V-module and $dw = 0$. The linear map

$$f : V \to W \tag{5.145}$$

$$v \mapsto v_{-1}w \tag{5.146}$$

is then a V-isomorphism.

Proof. By Theorems 5.26 and 5.27, f is a V-homomorphism. Since w generates W as a

V-module, f is surjective. Now let $v \in V$ be in the kernel of f, i.e. $f(v) = 0$. Then by Theorem 5.26, we have $Y(v, x)w = 0$, showing that $w \in \mathcal{M}(\{w\})$. Using the fact that w generates W as a V-module again, by Theorem 5.23 we have

$$W = \langle \{w\} \rangle = \mathcal{M}(\{v\}). \tag{5.147}$$

In other words, $Y(v, x)w = 0$ for all $w \in W$, and because W is faithful by hypothesis, it follows that $v = 0$. Therefore f is injective, and so is a V-isomorphism. $\square$

Chapter 6: Vertex operator algebras based on affine Lie algebras: $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ case

In this chapter we study a more complicated and physically interesting example of a vertex operator algebra that corresponds to a conformal field theory called the $\mathfrak{sl}_2(\mathbb{C})$ WZW model at non-critical level k.

6.1 The weak vertex algebra $\mathcal{E}(V)$, the weak vertex subalgebra $\mathcal{E}(V, d)$, and local subsets

Let V be a complex vector space. We define

$$\mathcal{E}(V) \equiv \mathrm{Hom}(V, V((x))). \tag{6.1}$$

Now we will define the notions of weak vertex operator and weak vertex algebra.

Definition 6.1 (Weak vertex operator)**.** A *weak vertex operator* on V is a formal series

$$a(x) = \sum_{n\in\mathbb{Z}} a_n x^{-n-1} \in (\mathrm{End}\ V)[[x, x^{-1}]] \tag{6.2}$$

such that $a_n \in \mathrm{End}\ V$ for all $n \in \mathbb{Z}$, and for every $v \in V$, we have $a(x)v \in V((x))$.

Definition 6.2. A *weak vertex algebra* is a vector space V together with a linear map

$$Y(\cdot, x) : V \to (\mathrm{End}\ V)[[x, x^{-1}]] \tag{6.3}$$

$$v \mapsto Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \tag{6.4}$$

for each formal variable x with $v_n \in \text{End } V$ for all $n \in \mathbb{Z}$, and a chosen element $\mathbf{1} \in V$. The following axioms must be satisfied: (1) the *vacuum property* $Y(\mathbf{1}, x) = \text{id}_V$, (2) the *creation property* $Y(v, x)\mathbf{1} \in V[[x]]$ and $\lim_{x \to 0} Y(v, x)\mathbf{1} = v$ for $v \in V$, and (3) the $\mathcal{D}$-bracket and derivative properties hold, that is,

$$[\mathcal{D}, Y(v, x)] = Y(\mathcal{D}v, x) = \frac{d}{dx} Y(v, x), \tag{6.5}$$

where $\mathcal{D} \in \text{End } V$ is defined by $\mathcal{D}(v) \equiv v_{-2}\mathbf{1}$ for $v \in V$.

Now we will consider the case $V = \mathbb{C}^2$ as an example.Then $\mathcal{E}(\mathbb{C}^2) = \text{Hom}(\mathbb{C}^2, \mathbb{C}^2((x)))$. The endomorphisms of $\mathbb{C}^2$ form the space of 2×2 matrices with complex coefficients, denoted $M_{2\times 2}(\mathbb{C})$. Now define $Y_{\mathcal{E}}(\cdot, x_0) : \mathcal{E}(\mathbb{C}^2) \to (\text{End } \mathcal{E}(\mathbb{C}^2))[[x_0, x_0^{-1}]]$ by

$$Y_{\mathcal{E}}(a(x), x_0)b(x) \equiv \text{Res}_{x_1} \left(x_0^{-1} \delta \left(\frac{x_1 - x}{x_0} \right) a(x_1)b(x) - x_0^{-1} \delta \left(\frac{x - x_1}{-x_0} \right) b(x)a(x_1) \right) \tag{6.6}$$

for $a(x), b(x) \in \mathcal{E}(\mathbb{C}^2)$. We also let $\mathbf{1} = \text{id}_{\mathbb{C}^2} = I \in \mathcal{E}(\mathbb{C}^2)$ where I is the 2×2 identity matrix. Now we will show that $(\mathcal{E}(\mathbb{C}^2), Y_{\mathcal{E}}, I)$ is a weak vertex algebra.

Theorem 6.3. $(\mathcal{E}(\mathbb{C}^2), Y_{\mathcal{E}}, I)$ is a weak vertex algebra. The operator $\mathcal{D}$ is the differentiation operator

$$\mathcal{D} = \frac{d}{dx} : M_{2\times 2}(\mathbb{C})[[x, x^{-1}]] \to M_{2\times 2}(\mathbb{C})[[x, x^{-1}]]. \tag{6.7}$$

Proof. First we check that $Y_{\mathcal{E}}(I, x_0)a(x) = a(x)$ for $a(x) \in \mathcal{E}(\mathbb{C}^2)$. By Theorems 1.31 and

1.32, we have

$$Y_{\mathcal{E}}(I, x_0)a(x) = \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)Ia(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)a(x)I\right) \tag{6.8}$$

$$= \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)a(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)a(x)\right) \tag{6.9}$$

$$= \mathrm{Res}_{x_1}\left(\left[x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right) - x_0^{-1}\delta\left(\frac{-x + x_1}{x_0}\right)\right]a(x)\right) \tag{6.10}$$

$$= \mathrm{Res}_{x_1}\left(x^{-1}\delta\left(\frac{x_1 - x_0}{x}\right)a(x)\right) \tag{6.11}$$

$$= \mathrm{Res}_{x_1}\left(x_1^{-1}\delta\left(\frac{x + x_0}{x_1}\right)a(x)\right) \tag{6.12}$$

$$= \mathrm{Res}_{x_1}\left(x_1^{-1}\delta\left(\frac{x + x_0}{x_1}\right)\right)a(x) = 1 \cdot a(x) = a(x). \tag{6.13}$$

Next we check that $Y_{\mathcal{E}}(a(x), x_0)I = e^{x_0\mathcal{D}}a(x)$. We have

$$Y_{\mathcal{E}}(a(x), x_0)I = \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)a(x_1)I - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)Ia(x_1)\right) \tag{6.14}$$

$$= \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)a(x_1) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)a(x_1)\right) \tag{6.15}$$

$$= \mathrm{Res}_{x_1}\left(\left[x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right) - x_0^{-1}\delta\left(\frac{-x + x_1}{x_0}\right)\right]a(x_1)\right) \tag{6.16}$$

$$= \mathrm{Res}_{x_1}\left(x^{-1}\delta\left(\frac{x_1 - x_0}{x}\right)a(x_1)\right) \tag{6.17}$$

$$= \mathrm{Res}_{x_1}\left(x_1^{-1}\delta\left(\frac{x + x_0}{x_1}\right)a(x_1)\right) \tag{6.18}$$

$$= \mathrm{Res}_{x_1}\left(x_1^{-1}\delta\left(\frac{x + x_0}{x_1}\right)a(x + x_0)\right) \tag{6.19}$$

$$= \mathrm{Res}_{x_1}\left(x_1^{-1}\delta\left(\frac{x + x_0}{x_1}\right)\right)a(x + x_0) = 1 \cdot a(x + x_0) = a(x + x_0) \tag{6.20}$$

$$= e^{x_0 d/dx}a(x) = e^{x_0\mathcal{D}}a(x). \tag{6.21}$$

It follows that $Y_{\mathcal{E}}(a(x), x_0)I \in \mathcal{E}(\mathbb{C}^2)[[x_0]]$ and that

$$\lim_{x_0 \to 0} Y_{\mathcal{E}}(a(x), x_0)I = a(x) \tag{6.22}$$

and

$$a(x)_{-2}I = \mathcal{D}(a(x)). \tag{6.23}$$

Finally, we need to show that

$$[\mathcal{D}, Y(v, x)] = Y(\mathcal{D}v, x) = \frac{d}{dx}Y(v, x). \tag{6.24}$$

Let $a(x), b(x) \in \mathcal{E}(\mathbb{C}^2)$. Then

$$\frac{\partial}{\partial x_0} Y_{\mathcal{E}}(a(x), x_0)b(x) \tag{6.25}$$

$$= \mathrm{Res}_{x_1} \left(\frac{\partial}{\partial x_0} \left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)\right) a(x_1)b(x) - \frac{\partial}{\partial x_0} \left(x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)\right) b(x)a(x_1)\right) \tag{6.26}$$

$$= \mathrm{Res}_{x_1} \left(\left(-\frac{\partial}{\partial x_1} x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)\right) a(x_1)b(x) + \left(\frac{\partial}{\partial x_1} x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)\right) b(x)a(x_1)\right) \tag{6.27}$$

$$= \mathrm{Res}_{x_1} \left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right) a'(x_1)b(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right) b(x)a'(x_1)\right) \tag{6.28}$$

$$= Y_{\mathcal{E}}(\mathcal{D}(a(x)), x_0)b(x) \tag{6.29}$$

and

$$[\mathcal{D}, Y_{\mathcal{E}}(a(x), x_0)]b(x) \tag{6.30}$$

$$= \mathcal{D}(Y_{\mathcal{E}}(a(x), x_0)b(x)) - Y_{\mathcal{E}}(a(x), x_0)\mathcal{D}(b(x)) \tag{6.31}$$

$$= \frac{\partial}{\partial x}(Y_{\mathcal{E}}(a(x), x_0)b(x)) - Y_{\mathcal{E}}(a(x), x_0)b'(x) \tag{6.32}$$

$$= \mathrm{Res}_{x_1} \left(\left(\frac{\partial}{\partial x} x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)\right) a(x_1)b(x) - \left(\frac{\partial}{\partial x} x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)\right) b(x)a(x_1)\right) \tag{6.33}$$

$$= \mathrm{Res}_{x_1} \left(\frac{\partial}{\partial x_0} \left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)\right) a(x_1)b(x) - \frac{\partial}{\partial x_0} \left(x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)\right) b(x)a(x_1)\right) \tag{6.34}$$

$$= \frac{\partial}{\partial x_0} Y_{\mathcal{E}}(a(x), x_0)b(x). \tag{6.35}$$

This completes the proof that $\mathcal{E}(\mathbb{C}^2)$ is a weak vertex algebra. □

Now given our complex vector space V, we define the subspace

$$\mathcal{E}(V, d) \equiv \{a(x) \in \mathcal{E}(V) | [d, a(x)] = a'(x)\} \subset \mathcal{E}(V) \tag{6.36}$$

where $d \in \text{End } V$ is an arbitrary linear operator on V. Returning to our example $V = \mathbb{C}^2$, we have $d \in M_{2\times 2}(\mathbb{C})$. Continuing with our example, we will show that $\mathcal{E}(\mathbb{C}^2, d)$ is a weak vertex subalgebra of $\mathcal{E}(\mathbb{C}^2)$.

Theorem 6.4. Let $d \in M_{2\times 2}(\mathbb{C})$ be an arbitrary linear operator on $\mathbb{C}^2$. Then the subspace $\mathcal{E}(\mathbb{C}^2, d)$ is a weak vertex subalgebra of $\mathcal{E}(\mathbb{C}^2)$.

Proof. Note that

$$[d, I] = dI - Id = d - d = 0 = \frac{d}{dx} I x^0, \tag{6.37}$$

so $I \in \mathcal{E}(\mathbb{C}^2, d)$. We want to show that

$$[d, Y_{\mathcal{E}}(a(x), x_0)b(x)] = \frac{\partial}{\partial x}(Y_{\mathcal{E}}(a(x), x_0)b(x)) \tag{6.38}$$

for $a(x), b(x) \in \mathcal{E}(\mathbb{C}^2, d)$. We have

$$\frac{\partial}{\partial x}(Y_{\mathcal{E}}(a(x), x_0)b(x)) \tag{6.39}$$

$$= \frac{\partial}{\partial x}\mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)a(x_1)b(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)b(x)a(x_1)\right) \tag{6.40}$$

$$= \mathrm{Res}_{x_1}\left(\left(\frac{\partial}{\partial x}x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)\right)a(x_1)b(x) - \left(\frac{\partial}{\partial x}x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)\right)b(x)a(x_1)\right) \tag{6.41}$$

$$+ \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)a(x_1)b'(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)b'(x)a(x_1)\right) \tag{6.42}$$

$$= -\mathrm{Res}_{x_1}\left(\left(\frac{\partial}{\partial x_1}x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)\right)a(x_1)b(x) - \left(\frac{\partial}{\partial x_1}x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)\right)b(x)a(x_1)\right) \tag{6.43}$$

$$+ \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)a(x_1)b'(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)b'(x)a(x_1)\right) \tag{6.44}$$

$$= \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)a'(x_1)b(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)b(x)a'(x_1)\right) \tag{6.45}$$

$$+ \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)a(x_1)b'(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)b'(x)a(x_1)\right) \tag{6.46}$$

$$= [d, Y_{\mathcal{E}}(a(x), x_0)b(x)]. \tag{6.47}$$

□

Using analogous proofs to the $V = \mathbb{C}^2$ case we get the following theorems.

Theorem 6.5. (Proposition 5.3.9 of [LL04]) Let V be a complex vector space. Then $(\mathcal{E}(V), Y_{\mathcal{E}}, \mathrm{id}_V)$ is a weak vertex algebra. Also,

$$\mathcal{D} = \frac{d}{dx} : (\mathrm{End}\ V)[[x, x^{-1}]] \to (\mathrm{End}\ V)[[x, x^{-1}]]. \tag{6.48}$$

Theorem 6.6. (Proposition 5.4.1 of [LL04]) Let $d \in$ End V be an arbitrary linear operator on V. Then the subspace $\mathcal{E}(V, d)$ of $\mathcal{E}(V)$ is a weak vertex subalgebra.

The following result immediately follows.

Theorem 6.7. (Theorem 5.4.2 of [LL04]) Let V be a vertex algebra. Let W be a vector space equipped with a linear operator d. Then the V-module structures (W, Y_W, d) are equivalent to the weak vertex algebra homomorphisms $V \to \mathcal{E}(W, d)$.

Next we will need the notion of a *local subset.*

Definition 6.8 (Local subset)**.** A subset or subspace A of $\mathcal{E}(V)$ is said to be *local* if for any $a(x), b(x) \in A$, $a(x)$ and $b(x)$ are *mutually local*, i.e. there exists a nonnegative integer k such that

$$B_k(x_1, -x_2)a(x_1)b(x_2) = B_k(x_1, -x_2)b(x_2)a(x_1). \tag{6.49}$$

Now we follow the part of the proof of Theorem 5.7.1 in [LL04] that we can for this example $V = \mathbb{C}^2$.

Theorem 6.9. Let $T \subset \mathbb{C}^2$ be any 1-dimensional subspace of $\mathbb{C}^2$. Define

$$Y_0(\cdot, x) : T \to \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2((x))) = \mathcal{E}(\mathbb{C}^2)$$

by

$$Y_0(\lambda e, x) \equiv \lambda I \tag{6.50}$$

where $\lambda \in \mathbb{C}$ and $T = \mathrm{span}\{e\}$. Define

$$T(x) \equiv \{Y_0(a, x) | a \in T\} \subset \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2((x))). \tag{6.51}$$

Define the linear operator

$$d \in \mathrm{End}\ \mathbb{C}^2 = M_{2\times 2}(\mathbb{C}). \tag{6.52}$$

Then $T(x)$ is a local subset and a vertex subalgebra of $\mathcal{E}(\mathbb{C}^2, d)$.

Proof. First we want to show that $T(x)$ is a local subset of $\mathcal{E}(\mathbb{C}^2, d)$. Let $a = \lambda e \in T$ and

$b = \mu e \in T$ be arbitrary elements in T. Then

$$[d, Y_0(a, x)] = d\lambda I - \lambda I d = \lambda d - \lambda d = 0 = \frac{d}{dx}(\lambda I) = \frac{d}{dx} Y_0(a, x) \tag{6.53}$$

and

$$B_0(x_1, -x_2)[Y_0(a, x_1), Y_0(b, x_2)] = [Y_0(a, x_1), Y_0(b, x_2)] = \lambda I \mu I - \mu I \lambda I = 0. \tag{6.54}$$

It then follows by definition that $T(x)$ is a local subset of $\mathcal{E}(\mathbb{C}^2, d)$.

Now we consider $a(x), b(x) \in T(x)$ where we use the notation from the book $a(x) \equiv Y_0(a, x)$ and define $Y_{\mathcal{E}}$ for such weak vertex operators (which will turn out to be vertex operators). Then:

$$Y_{\mathcal{E}}(a(x), x_0)b(x) = \operatorname{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)a(x_1)b(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)b(x)a(x_1)\right) \tag{6.55}$$

$$= \operatorname{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)\lambda I \mu I - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)\mu I \lambda I\right) \tag{6.56}$$

$$= \operatorname{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)\right)a(x)b(x) \tag{6.57}$$

$$= \operatorname{Res}_{x_1}\left(x^{-1}\delta\left(\frac{x_1 - x_0}{x}\right)\right)a(x)b(x) \tag{6.58}$$

$$= \operatorname{Res}_{x_1}\left(x_1^{-1}\delta\left(\frac{x + x_0}{x_1}\right)\right)a(x)b(x) \tag{6.59}$$

$$= \operatorname{Res}_{x_1}\left(x_1^{-1}\delta\left(\frac{x + x_0}{x_1}\right)\right)a(x)b(x) = 1 \cdot a(x)b(x) = a(x)b(x). \tag{6.60}$$

Now we want to show that $(T(x), Y_{\mathcal{E}}, I)$ is a vertex algebra. We have

$$Y_{\mathcal{E}}(a(x), x_0)b(x) = a(x)b(x) = \lambda\mu I \in \mathcal{E}(\mathbb{C}^2)((x_0)), \tag{6.61}$$

showing the truncation condition is satisfied. We also confirm that the vacuum property holds:

$$Y_{\mathcal{E}}(I, x) = I = \mathrm{id}_{\mathcal{E}(\mathbb{C}^2)}. \tag{6.62}$$

Next we check the creation property. We have

$$Y_{\mathcal{E}}(a(x), x_0)I = \lambda I^2 = \lambda I = a(x) \in \mathcal{E}(\mathbb{C}^2) \subset \mathcal{E}(\mathbb{C}^2)[[x_0]] \tag{6.63}$$

and

$$\lim_{x_0 \to 0} Y(a(x), x_0)I = \lim_{x_0 \to 0} a(x) = a(x), \tag{6.64}$$

showing the creation property holds. Lastly, we show that the Jacobi identity holds. We have

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(a(x), x_1)Y(b(x), x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(b(x), x_2)Y(a(x), x_1) \tag{6.65}$$

$$= x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) a(x)b(x) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) b(x)a(x) \tag{6.66}$$

$$= a(x)b(x)\left[x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\right] \tag{6.67}$$

and

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(a(x), x_0)b(x), x_2) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(a(x)b(x), x_2) \tag{6.68}$$

$$= a(x)b(x)x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right). \tag{6.69}$$

Equality of (6.64) and (6.66) follows immediately from the equation

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right), \tag{6.70}$$

stated in Theorem 1.30. □

Note that the vertex subalgebras of $\mathcal{E}(\mathbb{C}^2, d)$ constructed have the same structure as the $V = \mathbb{C}$ example we considered before, and are the only proper subalgebras of $\mathcal{E}(\mathbb{C}^2, d)$. Moreover, note that $T(x) \subset \mathcal{E}(\mathbb{C}^2, d)$ for all $d \in M_{2\times 2}(\mathbb{C})$ as shown in the proof.

6.2 Derivations and set up for the construction theorem

Recall that $\hat{\mathfrak{sl}}_2(\mathbb{C})$ is the central extension of the loop algebra

$$\hat{\mathfrak{sl}}_2(\mathbb{C}) \equiv \mathcal{L}(\mathfrak{sl}_2(\mathbb{C})) \oplus \mathbb{C}k = (\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}k \tag{6.71}$$

where k is a nonzero central element of $\hat{\mathfrak{sl}}_2(\mathbb{C})$. It is a Lie algebra with the bracket

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\mathrm{Tr}(ab)\delta_{m+n,0}k \tag{6.72}$$

for all $a, b \in \mathfrak{sl}_2(\mathbb{C})$ and $m, n \in \mathbb{Z}$. The previous section outlines an explicit computation of the bracket for a frequently used basis of $\hat{\mathfrak{sl}}_2(\mathbb{C})$.

The affine Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$ is equipped with the $\mathbb{Z}$-grading

$$\hat{\mathfrak{sl}}_2(\mathbb{C}) = \bigoplus_{n\in\mathbb{Z}} \hat{\mathfrak{sl}}_2(\mathbb{C})_{(n)}, \tag{6.73}$$

where $\hat{\mathfrak{sl}}_2(\mathbb{C})_{(0)} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}k$ and $\hat{\mathfrak{sl}}_2(\mathbb{C})_{(n)} = \mathfrak{sl}_2(\mathbb{C}) \otimes t^{-n}$ for $n \neq 0$. (6.70) makes $\hat{\mathfrak{sl}}_2(\mathbb{C})$ a $\mathbb{Z}$-graded Lie algebra. We also introduce two graded subalgebras of $\hat{\mathfrak{sl}}_2(\mathbb{C})$:

$$\hat{\mathfrak{sl}}_2(\mathbb{C})_{(\pm)} = \bigoplus_{n>0} \hat{\mathfrak{sl}}_2(\mathbb{C})_{(\pm n)} = \bigoplus_{n>0} \mathfrak{sl}_2(\mathbb{C}) \otimes t^{\mp n}, \tag{6.74}$$

$$\hat{\mathfrak{sl}}_2(\mathbb{C})_{(\leq 0)} = \bigoplus_{n\leq 0} \hat{\mathfrak{sl}}_2(\mathbb{C})_{(n)} = \hat{\mathfrak{sl}}_2(\mathbb{C})_{(-)} \oplus \hat{\mathfrak{sl}}_2(\mathbb{C})_{(0)} = \hat{\mathfrak{sl}}_2(\mathbb{C})_{(-)} \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}k. \tag{6.75}$$

As in [LL04], we will adopt the following notation.

Definition 6.10. For $a \in \mathfrak{sl}_2(\mathbb{C})$ we define the *generating function*

$$a(x) = \sum_{n \in \mathbb{Z}} (a \otimes t^n) x^{-n-1} \in \hat{\mathfrak{sl}}_2(\mathbb{C})[[x, x^{-1}]]. \tag{6.76}$$

Now we will define the vector space $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)$ and describe a $U(\hat{\mathfrak{sl}}_2(\mathbb{C})_{(\leq 0)})$ action on it that motivates the notation.

Definition 6.11. Let

$$V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0) \equiv U(\hat{\mathfrak{sl}}_2(\mathbb{C})) \otimes_{\mathbb{C}} \mathbb{C}_l \tag{6.77}$$

where $\mathbb{C}_l = \mathbb{C}$. We equip $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)$ with the following action of $U(\hat{\mathfrak{sl}}_2(\mathbb{C}))$. Let $\hat{\mathfrak{sl}}_2(\mathbb{C})_{(-)}$ act trivially on $\mathbb{C}_l$ and let k act as a scalar $l \in \mathbb{C}$, thus giving an action of $\hat{\mathfrak{sl}}_2(\mathbb{C})$ on $\mathbb{C}_l$. We then induce an action of $U(\hat{\mathfrak{sl}}_2(\mathbb{C})_{(\leq 0)})$ on $\mathbb{C}_l$ from it. We also have $U(\hat{\mathfrak{sl}}_2(\mathbb{C})_{(\leq 0)})$ act on $U(\hat{\mathfrak{sl}}_2(\mathbb{C}))$ by left multiplication. Thus we get an action of $U(\hat{\mathfrak{sl}}_2(\mathbb{C})_{(\leq 0)})$ on $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)$ by combining the actions of $U(\hat{\mathfrak{sl}}_2(\mathbb{C})_{(\leq 0)})$ on $\mathbb{C}_l$ and $U(\hat{\mathfrak{sl}}_2(\mathbb{C})_{(\leq 0)})$ on $U(\hat{\mathfrak{sl}}_2(\mathbb{C}))$ in the obvious way.

Now we will construct the linear operator $d : V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0) \to V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)$ to apply Theorem 5.7.1 of [LL04] in the main theorem of this chapter.

Definition 6.12. Define the linear operator $d_{(1)} : \hat{\mathfrak{sl}}_2(\mathbb{C}) \to \hat{\mathfrak{sl}}_2(\mathbb{C})$ by

$$d_{(1)}(k) = 0, \tag{6.78}$$

$$d_{(1)}(a \otimes t^n) = -n(a \otimes t^{n-1}) \forall a \in \mathfrak{sl}_2(\mathbb{C}), n \in \mathbb{Z}. \tag{6.79}$$

Then clearly $d_{(1)} \hat{\mathfrak{sl}}_2(\mathbb{C})_{(n)} \subset \hat{\mathfrak{sl}}_2(\mathbb{C})_{(n+1)}$ for $n \in \mathbb{Z}$, i.e. $d_{(1)}$ is an operator of weight one.

Lemma 6.13. The operator $d_{(1)} : \hat{\mathfrak{sl}}_2(\mathbb{C}) \to \hat{\mathfrak{sl}}_2(\mathbb{C})$ is a derivation of the Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$.

Proof. We want to show that

$$d_{(1)}([a \otimes t^m, b \otimes t^n]) = [d_{(1)}(a \otimes t^m), b \otimes t^n] + [a \otimes t^m, d_{(1)}(b \otimes t^n)] \tag{6.80}$$

for all $a, b \in \mathfrak{sl}_2(\mathbb{C})$ and $m, n \in \mathbb{Z}$. On the right hand side we have

$$[d_{(1)}(a \otimes t^m), b \otimes t^n] + [a \otimes t^m, d_{(1)}(b \otimes t^n)] \tag{6.81}$$

$$= [-m(a \otimes t^{m-1}), b \otimes t^n] + [a \otimes t^m, -n(b \otimes t^{n-1})] \tag{6.82}$$

$$= -m[a \otimes t^{m-1}, b \otimes t^n] - n[a \otimes t^m, b \otimes t^{n-1}] \tag{6.83}$$

$$= -m([a, b] \otimes t^{m+n-1} + (m-1)\mathrm{Tr}(ab)\delta_{m+n-1,0}k) \tag{6.84}$$

$$- n([a, b] \otimes t^{m+n-1} + m\mathrm{Tr}(ab)\delta_{m+n-1,0}k) \tag{6.85}$$

$$= -(m+n)[a, b] \otimes t^{m+n-1} + (-m(m-1) - mn)\mathrm{Tr}(ab)\delta_{m+n-1,0}k \tag{6.86}$$

$$= -(m+n)[a, b] \otimes t^{m+n-1} - m(m+n-1)\mathrm{Tr}(ab)\delta_{m+n-1,0}k \tag{6.87}$$

$$= -(m+n)[a, b] \otimes t^{m+n-1}. \tag{6.88}$$

On the left hand side we have

$$d_{(1)}([a \otimes t^m, b \otimes t^n]) = d_{(1)}([a, b] \otimes t^{m+n} + m\mathrm{Tr}(ab)\delta_{m+n,0}k) \tag{6.89}$$

$$= d_{(1)}([a, b] \otimes t^{m+n}) + d_{(1)}(m\mathrm{Tr}(ab)\delta_{m+n,0}k) \tag{6.90}$$

$$= d_{(1)}([a, b] \otimes t^{m+n}) + m\mathrm{Tr}(ab)\delta_{m+n,0}d_{(1)}(k) \tag{6.91}$$

$$= d_{(1)}([a, b] \otimes t^{m+n}) \tag{6.92}$$

$$= -(m+n)([a, b] \otimes t^{m+n-1}). \tag{6.93}$$

Therefore $d_{(1)}$ is a derivation on the Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$. □

Definition 6.14. Since we have shown that $d_{(1)}$ is a derivation of the Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$, it follows that $d_{(1)}$ naturally extends to a derivation of $U(\hat{\mathfrak{sl}}_2(\mathbb{C}))$. By the definition of $d_{(1)}$ we see that it preserves the subspace

$$\hat{\mathfrak{sl}}_2(\mathbb{C})_{(-)} \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}(k-l) \subset U(\hat{\mathfrak{sl}}_2(\mathbb{C})). \tag{6.94}$$

Since we can write $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0)$ as a left $U(\hat{\mathfrak{sl}}_2(\mathbb{C}))$-module

$$V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0) = U(\hat{\mathfrak{sl}}_2(\mathbb{C}))/U(\hat{\mathfrak{sl}}_2(\mathbb{C}))(\hat{\mathfrak{sl}}_2(\mathbb{C})_{(-)} \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}(k-l)), \tag{6.95}$$

the operator $d_{(1)}$ induces a linear operator $d : V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0) \to V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0)$.

Now we will prove a few lemmas in preparation for applying the construction theorem (Theorem 5.7.1 of [LL04]).

Lemma 6.15. Let $\phi : \mathfrak{sl}_2(\mathbb{C}) \to V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0)$ be the map that sends $a \mapsto a(-1)\mathbf{1} = a(-1)$. Define T to be the copy of $\mathfrak{sl}_2(\mathbb{C})$ equal to

$$T \equiv \phi(\mathfrak{sl}_2(\mathbb{C})) \subset V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0). \tag{6.96}$$

Then T is a local subset of $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0)$, i.e. for every $a, b \in T$ there exists a nonnegative integer k such that

$$B_k(x_1, -x_2)[a(x_1), b(x_2)] = 0. \tag{6.97}$$

Proof. Note that

$$[a(x_1), b(x_2)] = \sum_{m,n\in\mathbb{Z}} ([a,b] \otimes t^{m+n}) x_1^{-m-1} x_2^{-n-1} + \sum_{m\in\mathbb{Z}} m\mathrm{Tr}(ab) x_1^{-m-1} x_2^{m-1} k \tag{6.98}$$

$$= \sum_{m,n\in\mathbb{Z}} ([a,b] \otimes t^{m+n}) x_2^{-m-n-1} (x_1^{-m-1} x_2^{m}) + \sum_{m\in\mathbb{Z}} m\mathrm{Tr}(ab) x_1^{-m-1} x_2^{m-1} k \tag{6.99}$$

$$= [a,b](x_2) x_1^{-1} \delta\left(\frac{x_2}{x_1}\right) + \mathrm{Tr}(ab) \frac{\partial}{\partial x_2} x_1^{-1} \delta\left(\frac{x_2}{x_1}\right) k \tag{6.100}$$

$$= [a,b](x_2) x_2^{-1} \delta\left(\frac{x_1}{x_2}\right) - \mathrm{Tr}(ab) \frac{\partial}{\partial x_1} x_2^{-1} \delta\left(\frac{x_1}{x_2}\right) k. \tag{6.101}$$

We have

$$B_k(x_1, -x_2)[Y(a, x_1), Y(b, x_2)] = B_k(x_1, -x_2)[a(x_1), b(x_2)] \tag{6.102}$$

where k is a nonnegative integer. Choose k such that $a_n b \equiv (a \otimes t^n)b = 0$ for $n \geq k$ (as in the proof of Lemma 5.1). We then multiply the Jacobi identity by x_0^k and take Res_{x_0} to get

$$B_k(x_1, -x_2)[a(x_1), b(x_2)] = \mathrm{Res}_{x_0} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) x_0^k Y(Y(a, x_0)b, x_2) = 0. \tag{6.103}$$

Thus locality holds. □

Lemma 6.16. $Y(a,x)\mathbf{1}$ is an element of $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0)[[x]]$.

Proof. We have

$$Y(a,x)\mathbf{1} = a(x)1 = \left(\sum_{n\in\mathbb{Z}} (a \otimes t^n) x^{-n-1}\right) 1 = \sum_{n\in\mathbb{Z}} (a \otimes t^n) x^{-n-1} \tag{6.104}$$

$$= \sum_{n\leq -1} (a \otimes t^n) x^{-n-1} \in V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0)[[x]], \tag{6.105}$$

where to get the last equality we used the fact that

$$V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0) = U(\hat{\mathfrak{sl}}_2(\mathbb{C}))/U(\hat{\mathfrak{sl}}_2(\mathbb{C}))(\hat{\mathfrak{sl}}_2(\mathbb{C})_{(-)} \oplus \hat{\mathfrak{sl}}_2(\mathbb{C}) \oplus \mathbb{C}(k-l)) \tag{6.106}$$

as a left $U(\hat{\mathfrak{sl}}_2(\mathbb{C}))$-module (so a_n for $n > -1$ is killed by the quotient). □

Lemma 6.17.

$$\lim_{x\to 0} Y(a,x)\mathbf{1} = a. \tag{6.107}$$

Proof. We have

$$\lim_{x\to 0} Y(a,x)\mathbf{1} = \lim_{x\to 0} a(x)1 = \lim_{x\to 0} \sum_{n\in\mathbb{Z}} a(n)x^{-n-1} \tag{6.108}$$

$$= \lim_{x\to 0} \sum_{n\leq -1} a(n)x^{-n-1} = a(-1)x^0 \tag{6.109}$$

$$= a(-1) = a(-1)\mathbf{1}, \tag{6.110}$$

where we identify $a \otimes t^{-1}$ with a via the map $\phi : \mathfrak{sl}_2(\mathbb{C}) \to V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0)$ that sends $a \mapsto a(-1)\mathbf{1} = a(-1)$. □

Lemma 6.18. We have

$$[d, Y(a,x)] = \frac{d}{dx} Y(a,x). \tag{6.111}$$

Proof. We have

$$[d, a(n)] = -na(n-1), \tag{6.112}$$

so

$$[d, a(x)] = \left[d, \sum_{n\in\mathbb{Z}} a(n)x^{-n-1}\right] = \sum_{n\in\mathbb{Z}} [d, a(n)]x^{-n-1} \tag{6.113}$$

$$= \sum_{n\in\mathbb{Z}} -na(n-1)x^{-n-1}. \tag{6.114}$$

We also have

$$\frac{d}{dx}a(x) = \frac{d}{dx}\sum_{n\in\mathbb{Z}} a(n)x^{-n-1} = \sum_{n\in\mathbb{Z}} a(n)\frac{d}{dx}x^{-n-1} \tag{6.115}$$

$$= \sum_{n\in\mathbb{Z}} -(n+1)a(n)x^{-(n+2)} = \sum_{n\in\mathbb{Z}} -na(n-1)x^{-(n+1)} \tag{6.116}$$

$$= \sum_{n\in\mathbb{Z}} -na(n-1)x^{-n-1}. \tag{6.117}$$

Thus the result follows from $Y(a, x) = a(x)$. □

We have finished proving the preliminary results and are finally ready to proceed to the main theorem.

6.3 Maximal local subspaces of $\mathcal{E}(W)$ are vertex subalgebras

Here we show that maximal local subspaces of $\mathcal{E}(W)$ for a vector space W are vertex subalgebras.

Lemma 6.19. Let W be a vector space. Then 1_W is local with any weak vertex operator on W.

Proof. Let $a(x)$ be a weak vertex operator on W. Since $[1_W, a(x)] = 0$ due to $1_W \in \text{End}\, W$ being the identity operator, it immediately follows by the definition of locality of vertex operators that 1_W is local with $a(x)$. □

Lemma 6.20. (Proposition 5.2.2 of [LL04]) Let $a(x)$ and $b(x)$ be weak vertex operators on W. Then $a(x)_n b(x)$ is a weak vertex operator on W for every $n \in \mathbb{Z}$, and is given by

$$a(x)_n b(x) = \mathrm{Res}_{x_1}(B_n(x_1, -x)a(x_1)b(x) - B_n(-x, x_1)b(x)a(x_1)). \tag{6.118}$$

Proof. The formula immediately follows by the definition of $Y_{\mathcal{E}}(a(x), x_0)b(x)$ given by

$$Y_{\mathcal{E}}(a(x), x_0)b(x) \equiv \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right)a(x_1)b(x) - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right)b(x)a(x_1)\right) \tag{6.119}$$

and the notational choice for defining the coefficients $a(x)_n$ via

$$Y_{\mathcal{E}}(a(x), x_0)b(x) = \sum_{n\in\mathbb{Z}} a(x)_n b(x) x_0^{-n-1} \tag{6.120}$$

for $a(x), b(x) \in \mathcal{E}(W)$. Now we show that $a(x)_n b(x)$ is a weak vertex operator (or $a(x)_n b(x) \in \mathcal{E}(W)$). This follows immediately from

$$a(x)_n b(x) = \mathrm{Res}_{x_1}(B_n(x_1, -x)a(x_1)b(x) - B_n(-x, x_1)b(x)a(x_1)) \tag{6.121}$$

$$= \left(\sum_{i\geq 0}\binom{n}{i}(-x)^i a_{n-i}\right)b(x)w - b(x)\left(\sum_{i\geq 0}\binom{n}{i}(-x)^{n-i}a_i w\right) \tag{6.122}$$

and the observation that $a(x)_n b(x)w \in W((x))$ because

$$\sum_{i\geq 0}\binom{n}{i}(-x)^{n-i}a_i w \tag{6.123}$$

is a finite sum and $b(x)w \in W((x))$. □

Theorem 6.21. (Theorem 5.5.11 of [LL04]) Let V be a vertex subalgebra of $\mathcal{E}(W)$. Then V is a local subalgebra and W is a faithful module of the vertex algebra V with

$$Y_W(a(x), x_0) = a(x_0) \tag{6.124}$$

for $a(x) \in V$.

Proof. Note that $V \subset \mathcal{E}(W)$. Then for $a(x), b(x) \in V$ and $w \in W$, by the definition of $Y_{\mathcal{E}}(a(x), x_0)b(x)$ we have

$$(Y_{\mathcal{E}}(a(x), x_0)b(x))w \tag{6.125}$$

$$= \mathrm{Res}_{x_1} \left(x_0^{-1}\delta\left(\frac{x_1 - x}{x_0}\right) a(x_1)b(x)w - x_0^{-1}\delta\left(\frac{x - x_1}{-x_0}\right) b(x)a(x_1)w \right). \tag{6.126}$$

Then by Corollary 5.16, W is a V-module and $Y_W(a(x), x_0) = a(x_0)$. By Lemma 5.14, weak commutativity holds for Y_W, which is clearly equivalent to the statement that any pair $a(x), b(x) \in V$ are mutually local. This shows that V is a local subalgebra. □

Lemma 6.22. (Proposition 5.5.12 of [LL04]) Let $a(x), b(x) \in \mathcal{E}(W)$ be mutually local and $k \geq 0$ so that

$$B_k(x_1, -x_2)a(x_1)b(x_2)w = B_k(x_1, -x_2)b(x_2)a(x_1)w \tag{6.127}$$

for all $w \in W$. Then we have

$$a(x)_n b(x) = 0 \tag{6.128}$$

for $n \geq k$. Also, if V is a local subalgebra of $\mathcal{E}(W)$ then given $a(x), b(x) \in V$ we have $a(x)_n b(x) = 0$ for sufficiently large n.

Proof. This follows immediately from Lemma 6.20 and the condition

$$B_k(x_1, -x_2)a(x_1)b(x_2)w = B_k(x_1, -x_2)b(x_2)a(x_1)w \tag{6.129}$$

for all $w \in W$. $\square$

Lemma 6.23. (Proposition 5.5.13 of [LL04]) Let $a(x), b(x) \in \mathcal{E}(W)$ be mutually local with $k \geq 0$ so that

$$B_k(x_1, -x_2)a(x_1)b(x_2)w = B_k(x_1, -x_2)b(x_2)a(x_1)w \tag{6.130}$$

for all $w \in W$. Then we have

$$B_k(x_1, -x_2)Y_{\mathcal{E}}(a(x), x_1)Y_{\mathcal{E}}(b(x), x_2)c(x) = B_k(x_1, -x_2)Y_{\mathcal{E}}(b(x), x_2)Y_{\mathcal{E}}(a(x), x_1)c(x) \tag{6.131}$$

for all $c(x) \in \mathcal{E}(W)$. Also, if V is a local subalgebra of $\mathcal{E}(W)$ then weak commutativity holds for $Y_{\mathcal{E}}$ on V.

Proof. Let $c(x) \in \mathcal{E}(W)$. By definition of $Y_{\mathcal{E}}$,

$$Y_{\mathcal{E}}(a(x), x_1)Y_{\mathcal{E}}(b(x), x_2)c(x) \tag{6.132}$$

$$= \mathrm{Res}_{x_3} x_1^{-1}\delta\left(\frac{x_3 - x}{x_1}\right) a(x_3)(Y_{\mathcal{E}}(b(x), x_2)c(x)) \tag{6.133}$$

$$- x_1^{-1}\delta\left(\frac{x - x_3}{-x_1}\right) (Y_{\mathcal{E}}(b(x), x_2)c(x))a(x_3) \tag{6.134}$$

$$= \mathrm{Res}_{x_3}\mathrm{Res}_{x_4} x_1^{-1}\delta\left(\frac{x_3 - x}{x_1}\right) x_2^{-1}\delta\left(\frac{x_4 - x}{x_2}\right) a(x_3)b(x_4)c(x) \tag{6.135}$$

$$- \mathrm{Res}_{x_3}\mathrm{Res}_{x_4} x_1^{-1}\delta\left(\frac{x_3 - x}{x_1}\right) x_2^{-1}\delta\left(\frac{x - x_4}{-x_2}\right) a(x_3)c(x)b(x_4) \tag{6.136}$$

$$- \mathrm{Res}_{x_3}\mathrm{Res}_{x_4} x_1^{-1}\delta\left(\frac{x - x_3}{-x_1}\right) x_2^{-1}\delta\left(\frac{x_4 - x}{x_2}\right) b(x_4)c(x)a(x_3) \tag{6.137}$$

$$- \mathrm{Res}_{x_3}\mathrm{Res}_{x_4} x_1^{-1}\delta\left(\frac{x - x_3}{-x_1}\right) x_2^{-1}\delta\left(\frac{x - x_4}{-x_2}\right) c(x)b(x_4)a(x_3). \tag{6.138}$$

Using the same approach,

$$Y_{\mathcal{E}}(b(x),x_2)Y_{\mathcal{E}}(a(x),x_1)c(x) \tag{6.139}$$

$$= \mathrm{Res}_{x_3}\mathrm{Res}_{x_4}x_1^{-1}\delta\left(\frac{x_3-x}{x_1}\right)x_2^{-1}\delta\left(\frac{x_4-x}{x_2}\right)b(x_4)a(x_3)c(x) \tag{6.140}$$

$$-\mathrm{Res}_{x_3}\mathrm{Res}_{x_4}x_1^{-1}\delta\left(\frac{x_3-x}{x_1}\right)x_2^{-1}\delta\left(\frac{x-x_4}{-x_2}\right)b(x_4)c(x)a(x_3) \tag{6.141}$$

$$-\mathrm{Res}_{x_3}\mathrm{Res}_{x_4}x_1^{-1}\delta\left(\frac{x-x_3}{-x_1}\right)x_2^{-1}\delta\left(\frac{x_4-x}{x_2}\right)a(x_3)c(x)b(x_4) \tag{6.142}$$

$$-\mathrm{Res}_{x_3}\mathrm{Res}_{x_4}x_1^{-1}\delta\left(\frac{x-x_3}{-x_1}\right)x_2^{-1}\delta\left(\frac{x-x_4}{-x_2}\right)c(x)a(x_3)b(x_4). \tag{6.143}$$

Then by the previous result

$$x_1^{-1}\delta\left(\frac{x_2-y}{x_1}\right)f(x_1,x_2,y) = x_1^{-1}\delta\left(\frac{x_2-y}{x_1}\right)f(x_2-y,x_2,y) \tag{6.144}$$

for $f(x_1,x_2,y) \in (\mathrm{End}\ V)[[x_1,x_1^{-1},x_2,x_2^{-1},y,y^{-1}]]$, we obtain

$$B_k(x_1,-x_2)x_1^{-1}\delta\left(\frac{x_3-x}{x_1}\right)x_2^{-1}\delta\left(\frac{x_4-x}{x_2}\right) \tag{6.145}$$

$$= B_k(x_3,-x_4)x_1^{-1}\delta\left(\frac{x_3-x}{x_1}\right)x_2^{-1}\delta\left(\frac{x_4-x}{x_2}\right) \tag{6.146}$$

and similar identities for other products of the formal delta functions. This shows that

$$B_k(x_1,-x_2)Y_{\mathcal{E}}(a(x),x_1)Y_{\mathcal{E}}(b(x),x_2)c(x) = B_k(x_1,-x_2)Y_{\mathcal{E}}(b(x),x_2)Y_{\mathcal{E}}(a(x),x_1)c(x) \tag{6.147}$$

for all $c(x) \in \mathcal{E}(W)$ using the assumption

$$B_k(x_1,-x_2)a(x_1)b(x_2)w = B_k(x_1,-x_2)b(x_2)a(x_1)w \tag{6.148}$$

for all $w \in W$ along with the expressions we found for $Y_{\mathcal{E}}(a(x), x_1)Y_{\mathcal{E}}(b(x), x_2)c(x)$ and $Y_{\mathcal{E}}(b(x), x_2)Y_{\mathcal{E}}(a(x), x_1)c(x)$. □

Theorem 6.24. (Theorem 5.5.14 of [LL04]) Any local subalgebra V of $\mathcal{E}(W)$ is a vertex algebra and W is a faithful module, where $Y_W(a(x), x_0) = a(x_0)$ for $a(x) \in V$. In particular, the local subalgebras of $\mathcal{E}(W)$ are precisely the vertex subalgebras.

Proof. That V is a vertex algebra follows from Theorems 5.11 and 6.5, and Lemmas 6.22 and 6.23. By Theorem 6.21 W is a module meeting the stated conditions. □

Lemma 6.25. (Proposition 5.5.15 of [LL04]) Let $a(x)$, $b(x)$, and $c(x)$ be pairwise mutually local weak vertex operators on W. Then $a(x)_n b(x)$ and $c(x)$ are mutually local for all $n \in \mathbb{Z}$.

Proof. Let $n \in \mathbb{Z}$. Let r be a nonnegative integer such that $r \geq -n$ and

$$B_r(x_1, -x_2)a(x_1)b(x_2) = B_r(x_1, -x_2)b(x_2)a(x_1), \tag{6.149}$$

$$B_r(x_1, -x_2)a(x_1)c(x_2) = B_r(x_1, -x_2)c(x_2)a(x_1), \tag{6.150}$$

$$B_r(x_1, -x_2)b(x_1)c(x_2) = B_r(x_1, -x_2)c(x_2)b(x_1). \tag{6.151}$$

We have

$$a(x)_n b(x) = \mathrm{Res}_{x_1}(B_n(x_1, -x)a(x_1)b(x) - B_n(-x, x_1)b(x)a(x_1)). \tag{6.152}$$

Since $B_{4r}(x, -x_2) = B_{3r}(x - x_1, x_1 - x_2)B_r(x, -x_2)$, by the conditions on r we have

$$B_{4r}(x, -x_2)(B_n(x_1, -x)a(x_1)b(x)c(x_2) - B_n(-x, x_1)b(x)a(x_1)c(x_2)) \tag{6.153}$$

$$= \sum_{s=0}^{3r} \binom{3r}{s} B_{3r-s}(x, -x_1)B_s(x_1, -x_2)B_r(x, -x_2) \tag{6.154}$$

$$(B_n(x_1, -x)a(x_1)b(x)c(x_2) - B_n(-x, x_1)b(x)a(x_1)c(x_2)) \tag{6.155}$$

$$= \sum_{s=r+1}^{3r} \binom{3r}{s} B_{3r-s}(x, -x_1)B_s(x_1, -x_2)B_r(x, -x_2) \tag{6.156}$$

$$(B_n(x_1, -x)a(x_1)b(x)c(x_2) - B_n(-x, x_1)b(x)a(x_1)c(x_2)) \tag{6.157}$$

$$= \sum_{s=r+1}^{3r} \binom{3r}{s} B_{3r-s}(x, -x_1)B_s(x_1, -x_2)B_r(x, -x_2) \tag{6.158}$$

$$(B_n(x_1, -x)c(x_2)a(x_1)b(x) - B_n(-x, x_1)c(x_2)b(x)a(x_1)) \tag{6.159}$$

$$= \sum_{s=0}^{3r} \binom{3r}{s} B_{3r-s}(x, -x_1)B_s(x_1, -x_2)B_r(x, -x_2) \tag{6.160}$$

$$(B_n(x_1, -x)c(x_2)a(x_1)b(x) - B_n(-x, x_1)c(x_2)b(x)a(x_1)) \tag{6.161}$$

$$= B_{4r}(x, -x_2)(B_n(x_1, -x)c(x_2)a(x_1)b(x) - B_n(-x, x_1)c(x_2)b(x)a(x_1)). \tag{6.162}$$

Applying Res_{x_1} to the equation found above results in

$$B_{4r}(x, -x_2)(a(x)_n b(x))c(x_2) = B_{4r}(x, -x_2)c(x_2)(a(x)_n b(x)), \tag{6.163}$$

where the mutual locality of $a(x)_n b(x)$ and $c(x)$ immediately follows by definition. □

Lemma 6.26. (Theorem 5.5.17 of [LL04]) Let W be a vector space. Any maximal local subspace of $\mathcal{E}(W)$ is a vertex subalgebra with W as a faithful module.

Proof. We follow the proof of Theorem 5.5.17 of [LL04, 172]. Let A be a maximal local subspace of $\mathcal{E}(W)$. Note that 1_W is local with any weak vertex operator on W by the previous lemma. Thus $A + \mathbb{C}1_W$ is a local subspace of $\mathcal{E}(W)$. Since A was taken to be maximal, we have $A = A + \mathbb{C}1_W$, and so $1_W \in A$.

Now let $a(x), b(x) \in A$ and $n \in \mathbb{Z}$. By Lemma 6.25, $a(x)_n b(x)$ is local with $a(x)$ and $b(x)$ (more generally, any vertex operator in A). Using Lemma 6.25 again, $a(x)_n b(x)$ is local with itself. Then using a similar argument to the first paragraph, $A + \mathbb{C}a(x)_n b(x)$ is a local subspace of $\mathcal{E}(W)$, and so $A = A + \mathbb{C}a(x)_n b(x)$, resulting in $a(x)_n b(x) \in A$. This shows that A is a weak vertex subalgebra of $\mathcal{E}(W)$. Therefore A is a local subalgebra, and by Theorem 6.24, a vertex algebra with W as a module. □

6.4 Generating vertex subalgebras of $\mathcal{E}(W)$ by pairwise mutually local vertex operators on W

Lemma 6.27. (Theorem 5.5.18 of [LL04]) Let S be a set of pairwise mutually local vertex operators on W, i.e., a local subset of $\mathcal{E}(W)$. Then S can be embedded in a vertex subalgebra of $\mathcal{E}(W)$, and in fact, the weak vertex subalgebra $\langle S \rangle$ generated by S is a vertex algebra, with W as a natural faithful module. Furthermore, $\langle S \rangle$ is the linear span of the elements of the form

$$a^{(1)}(x)_{n_1} \cdots a^{(r)}(x)_{n_r} 1_W \tag{6.164}$$

for $a^{(i)}(x) \in S$, $n_1, \ldots, n_r \in \mathbb{Z}$, with $r \geq 0$. In particular, this linear span carries the structure of a vertex algebra.

Proof. It follows from Zorn's lemma that there exists a maximal local subspace V containing S. By Lemma 6.26, V is a vertex subalgebra, with W as a natural module. Since $\langle S \rangle$ is a weak subalgebra of V, $\langle S \rangle$ is necessarily a vertex subalgebra and W is an $\langle S \rangle$-module. The rest follows immediately from Theorem 5.13. □

6.5 The construction theorem and vertex algebra proof

Now we will show that $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0)$ is a vertex algebra using the proof of Theorem 5.7.1 of [LL04, 179-181].

Lemma 6.28. $T(x)$ is a local subset of $\mathcal{E}(V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0),d)$.

Proof. Recall from a previous section of this chapter that we have shown

$$[d,a(x)] = \frac{d}{dx}a(x) \tag{6.165}$$

and

$$B_k(x_1,-x_2)[a(x_1),b(x_2)] = 0 \tag{6.166}$$

for some nonnegative integer k, where $a,b \in T$. It follows by definition that $T(x)$ is a local subset of $\mathcal{E}(V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0),d)$. $\square$

Lemma 6.29. $\langle T(x)\rangle$ is a vertex subalgebra of $\mathcal{E}(V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0),d)$ with $(V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0),d)$ as a module.

Proof. We follow the proof of Lemma 6.27. By Zorn's lemma there exists a maximal local subspace V containing $T(x)$. It immediately follows that $V = \langle T(x)\rangle$ is a vertex subalgebra with $(V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0),d)$ as a module. $\square$

Theorem 6.30. (Theorem 5.6.1 of [LL04]) Let S be a local subset of $\mathcal{E}(W,d)$. Then the weak vertex subalgebra $\langle S\rangle$ of $\mathcal{E}(W)$ is a vertex subalgebra of $\mathcal{E}(W,d)$ where (W,d) is a faithful module of the vertex algebra $\langle S\rangle$ and $Y_W(a(x),x_0) = a(x_0)$ for all $a(x) \in \langle S\rangle$. Moreover, the d-bracket formula

$$Y_W(\mathcal{D}a(x),x_0) = [d, Y_W(a(x),x_0)] \tag{6.167}$$

holds for all $a(x) \in \langle S\rangle$.

Proof. By Lemma 6.27, $\langle S\rangle$ is a vertex subalgebra of $\mathcal{E}(W)$ where W is a module of $\langle S\rangle$. Because $S\subset\mathcal{E}(W,d)$ and by Theorem 6.6 $\mathcal{E}(W,d)$ is a weak vertex subalgebra of $\mathcal{E}(W)$, $\langle S\rangle$ is a vertex subalgebra of $\mathcal{E}(W,d)$. We obtain the rest at once by Theorem 6.7. $\square$

Now we are ready to prove the construction theorem that will show $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l,0)$ is a vertex algebra.

Theorem 6.31. (Theorem 5.7.1 of [LL04]) Let V be a vector space equipped with a vector $\mathbf{1}$ and linear operator d so that $d\mathbf{1}=0$. Let $T\subset V$ be equipped with a map

$$Y_0(\cdot,x):T\to\mathrm{Hom}(V,V((x))) \tag{6.168}$$

$$a\mapsto Y_0(a,x)=\sum_{n\in\mathbb{Z}}a_nx^{-n-1}. \tag{6.169}$$

Now assume that for $a\in T$ we have $Y_0(a,x)\mathbf{1}\in V[[x]]$ and

$$\lim_{x\to 0}Y_0(a,x)\mathbf{1}=a, \tag{6.170}$$

$$[d,Y_0(a,x)]=\frac{d}{dx}Y_0(a,x). \tag{6.171}$$

Further assume that for $a,b\in T$ there exists a nonnegative integer k so that

$$B_k(x_1,-x_2)[Y_0(a,x_1),Y_0(b,x_2)]=0 \tag{6.172}$$

and V is spanned by the vectors

$$a^{(1)}_{n_1}\cdots a^{(r)}_{n_r}\mathbf{1} \tag{6.173}$$

for $r\geq 0$, $a^{(i)}\in T$, and $n_i\in\mathbb{Z}$ as a vector space. Then we can extend Y_0 uniquely to a linear map $Y:V\to\mathrm{Hom}(V,V((x)))$ so that $(V,Y,\mathbf{1})$ is a vertex algebra. Moreover, Y is given by

$$Y(a^{(1)}_{n_1}\cdots Y(a^{(r)}_{n_r}\mathbf{1},x)=a^{(1)}(x)_{n_1}\cdots a^{(r)}(x)_{n_r}\mathrm{id}_V, \tag{6.174}$$

where we define the notation

$$a(x) \equiv Y_0(a, x) \tag{6.175}$$

for $a \in T$. Also, the operator $d = \mathcal{D}$ on V. Finally, if we define

$$T(x) \equiv \{a(x)|a \in T\} \subset \mathrm{Hom}(V, V((x))) \tag{6.176}$$

then the linear map

$$\psi : \langle T(x) \rangle \to V \tag{6.177}$$

$$\alpha(x) \mapsto \mathrm{Res}_x x^{-1} \alpha(x) \mathbf{1} \tag{6.178}$$

is a vertex algebra isomorphism, where we note that $\langle T(x) \rangle$ is the vertex algebra constructed in Lemma 6.27.

Proof. Uniqueness and the formula for Y follow by the spanning hypothesis. By

$$[d, Y_0(a, x)] = \frac{d}{dx} Y_0(a, x) \tag{6.179}$$

and

$$B_k(x_1, -x_2)[Y_0(a, x_1), Y_0(b, x_2)] = 0 \tag{6.180}$$

we conclude that $T(x)$ is a local subset of $\mathcal{E}(V, d)$. Then by Theorem 6.30 we conclude that $\langle T(x) \rangle$ is a vertex subalgebra of $\mathcal{E}(W, d)$ where (W, d) is a module equipped with an action $Y_V(\alpha(x), x_0) = \alpha(x_0)$ for $\alpha(x) \in \langle T(x) \rangle$. Note that

$$\psi(\alpha(x)) = \mathrm{Res}_{x_0} x_0^{-1} \alpha(x_0) \mathbf{1} = \mathrm{Res}_{x_0} x_0^{-1} Y_V(\alpha(x), x_0) \mathbf{1}. \tag{6.181}$$

Then by Theorem 5.28 it follows that ψ is a $\langle T(x) \rangle$-module isomorphism because (V, d) is a faithful $\langle T(x) \rangle$-module, $d\mathbf{1} = 0$, and V is spanned by elements of the form $a^{(1)}_{n_1} \cdots a^{(r)}_{n_r} \mathbf{1}$

for $r \geq 0$, $a^{(i)} \in T$, and $n_i \in \mathbb{Z}$. Therefore we induce a vertex algebra structure on V via $\psi(\langle T(x)\rangle) = V$. We find that $\mathbf{1}$ is the vacuum vector for V since

$$\psi(\mathrm{id}_V) = \mathrm{Res}_{x_0} x_0^{-1} Y_V(\mathrm{id}_V, x_0)\mathbf{1} = \mathrm{Res}_{x_0} x_0^{-1} \mathrm{id}_V(x_0)\mathbf{1} = \mathrm{id}_V \mathbf{1} = \mathbf{1}. \tag{6.182}$$

Now by $Y(\cdot, x)$ we denote the vertex operator map for V. Let $a \in T$. By

$$\lim_{x \to 0} Y_0(a, x)\mathbf{1} = a \tag{6.183}$$

we have

$$\psi(a(x)) = \lim_{x \to 0} a(x)\mathbf{1} = a. \tag{6.184}$$

Since ψ is a $\langle T(x)\rangle$-module homomorphism, we have

$$Y(a, x_0) = \psi Y_{\mathcal{E}}(\psi^{-1}a, x_0)\psi^{-1} = Y_V(\psi^{-1}a, x_0) = Y_V(a(x), x_0) = a(x_0). \tag{6.185}$$

This shows that Y is indeed an extension of Y_0. Since V is spanned by the coefficients of all the monomials in the expressions

$$Y(a^{(1)}, x_1)Y(a^{(2)}, x_2) \cdots Y(a^{(r)}, x_r)\mathbf{1} \tag{6.186}$$

for $a^{(i)} \in T$, by the $\mathcal{D}$-bracket derivative formula, the d-bracket derivative formula given in the hypothesis, and $\mathcal{D}\mathbf{1} = 0 = d\mathbf{1}$, we conclude that d and $\mathcal{D}$ act the same way on the expressions

$$Y(a^{(1)}, x_1)Y(a^{(2)}, x_2) \cdots Y(a^{(r)}, x_r)\mathbf{1}. \tag{6.187}$$

This completes the proof. □

As a corollary, we have the following main result.

Corollary 6.32. (Theorem 6.2.11 of [LL04]) Let $l \in \mathbb{C}$. Then there exists a unique vertex algebra structure $(V_{\hat{\mathfrak{sl}_2(\mathbb{C})}}(l,0), Y, \mathbf{1})$ on $V_{\hat{\mathfrak{sl}_2(\mathbb{C})}}(l,0)$ so that $\mathbf{1} = 1 \in \mathbb{C}$ is the vacuum vector and

$$Y(a,x) \equiv a(x) \in (\text{End } V_{\hat{\mathfrak{sl}_2(\mathbb{C})}}(l,0))[[x,x^{-1}]] \tag{6.188}$$

for $a \in \mathfrak{sl}_2(\mathbb{C})$. In particular, Y is given by

$$Y(a^{(1)}(n_1)\cdots a^{(r)}(n_r)\mathbf{1}, x) = a^{(1)}(x)_{n_1}\cdots a^{(r)}(x)_{n_r}\text{id}_{V_{\hat{\mathfrak{sl}_2(\mathbb{C})}}(l,0)} \tag{6.189}$$

for $r \geq 0$, $a^{(i)} \in \mathfrak{sl}_2(\mathbb{C})$, and $n_i \in \mathbb{Z}$.

Proof. Note that $V_{\hat{\mathfrak{sl}_2(\mathbb{C})}}(l,0)$ is a restricted $\hat{\mathfrak{sl}_2(\mathbb{C})}$-module. The relations for the commutator $[a(x_1), b(x_2)]$ derived before imply locality. The existence and uniqueness of the vertex algebra structure follows immediately from Theorem 6.31 with the linear operator d we constructed earlier and setting $V = V_{\hat{\mathfrak{sl}_2(\mathbb{C})}}(l,0)$, $T = \mathfrak{sl}_2(\mathbb{C}) \subset V$, and $Y_0(a,x) = a(x)$ for $a \in \mathfrak{sl}_2(\mathbb{C})$, along with the lemmas showing the conditions for applying the theorem are met. □

6.6 $V_{\hat{\mathfrak{sl}_2(\mathbb{C})}}(l,0)$ is a vertex operator algebra

Lemma 6.33. (Theorem 6.2.16 of [LL04]) For $a \in \mathfrak{sl}_2(\mathbb{C})$ and $m, n \in \mathbb{Z}$, we have

$$[L(m), a(n)] = -na(m+n), \tag{6.190}$$

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\frac{dl}{l+h}\delta_{m+n,0} \tag{6.191}$$

on $V_{\hat{\mathfrak{sl}_2(\mathbb{C})}}(l,0)$. Also, on $V_{\hat{\mathfrak{sl}_2(\mathbb{C})}}(l,0)$ we have $L(0)v = nv$ for $v \in V_{\hat{\mathfrak{sl}_2(\mathbb{C})}}(l,0)_{(n)}$ where $n \geq 0$ and $L(-1) = \mathcal{D}$ where $\mathcal{D}$ is the $\mathcal{D}$-operator of the vertex algebra $V_{\hat{\mathfrak{sl}_2(\mathbb{C})}}(l,0)$.

Proof. We have

$$[a(m), L(n)] = ma(m+n) \tag{6.192}$$

for $a \in \mathfrak{sl}_2(\mathbb{C})$ and $m, n \in \mathbb{Z}$. We then obtain

$$[Y(a, x_1), Y(\omega, x_2)] = -a(x_2)\frac{\partial}{\partial x_1} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right). \tag{6.193}$$

By the commutator formula

$$[Y(u, x_1), Y(v, x_2)] = \sum_{i=0}^{n} \frac{(-1)^i}{i!} Y(u_i v, x_2)\left(\frac{\partial}{\partial x_1}\right)^i x_2^{-1}\delta\left(\frac{x_1}{x_2}\right), \tag{6.194}$$

it is enough to show that

$$a_n\omega = a(n)\omega = \delta_{n,1} a \tag{6.195}$$

for $n \geq 0$. Because $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)$ is $\mathbb{Z}$-graded as a $\hat{\mathfrak{sl}}_2(\mathbb{C})$-module with $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)_{(n)} = 0$ for $n < 0$ and $\omega \in V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)_{(2)}$, we have $a(m)\omega = 0$ for $m > 2$. We also compute

$$2(l+h)a(2)\omega = \sum_{i=1}^{d} a(2)u^{(i)}(-1)^2\mathbf{1} \tag{6.196}$$

$$= \sum_{i=1}^{d} ([a, u^{(i)}](1)u^{(i)}(-1)\mathbf{1} + u^{(i)}(-1)a(2)u^{(i)}(-1)\mathbf{1}) \tag{6.197}$$

$$= \sum_{i=1}^{d} l\langle [a, u^{(i)}], u^{(i)}\rangle \mathbf{1} \tag{6.198}$$

$$= \sum_{i=1}^{d} l\langle a, [u^{(i)}, u^{(i)}]\rangle \mathbf{1} = 1 \tag{6.199}$$

where $\langle \cdot, \cdot \rangle : \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C}$ is the Cartan-Killing form defined by $\langle a, b\rangle \equiv \mathrm{Tr}(ab)$ for all $a, b \in \mathfrak{sl}_2(\mathbb{C})$ and we used the fact that $b(m)\mathbf{1} = 0$ for $b \in \mathfrak{sl}_2(\mathbb{C})$ and $m \geq 0$. By a similar calculation we have

$$2(l+h)a(1)\omega \tag{6.200}$$

$$= \sum_{i=1}^{d}([a,u^{(i)}](0)u^{(i)}(-1)\mathbf{1} + l\langle a,u^{(i)}\rangle u^{(i)}(-1)\mathbf{1} + u^{(i)}(-1)a(1)u^{(i)}(-1)\mathbf{1}) \tag{6.201}$$

$$= \sum_{i=1}^{d}([[a,u^{(i)}],u^{(i)}](-1)\mathbf{1} + 2l\langle a,u^{(i)}\rangle u^{(i)}(-1)\mathbf{1}) \tag{6.202}$$

$$= 2ha(-1)\mathbf{1} + 2la(-1)\mathbf{1} \tag{6.203}$$

$$= 2(l+h)a(-1)\mathbf{1} = 2(l+h)a, \tag{6.204}$$

where we used the fact that the Casimir element Ω acts like the scalar $2h$ on $\mathfrak{sl}_2(\mathbb{C})$. We also have

$$2(l+h)a(0)\omega = \sum_{i=1}^{d}([a,u^{(i)}](-1)u^{(i)}(-1)\mathbf{1} + u^{(i)}(-1)[a,u^{(i)}](-1)\mathbf{1}) \tag{6.205}$$

$$= \sum_{1\le i,j\le d}\langle [a,u^{(i)}],u^{(j)}\rangle u^{(j)}(-1)u^{(i)}(-1)\mathbf{1} \tag{6.206}$$

$$+ \sum_{1\le i,j\le d}\langle [a,u^{(i)}],u^{(j)}\rangle u^{(i)}(-1)u^{(j)}(-1)\mathbf{1} \tag{6.207}$$

$$= \sum_{1\le i,j\le d}(\langle [a,u^{(i)}],u^{(j)}\rangle + \langle [a,u^{(j)}],u^{(i)}\rangle)u^{(i)}(-1)u^{(j)}(-1)\mathbf{1} \tag{6.208}$$

$$= \sum_{1\le i,j\le d}(\langle a,[u^{(i)},u^{(j)}]\rangle + \langle a,[u^{(j)},u^{(i)}]\rangle)u^{(i)}(-1)u^{(j)}(-1)\mathbf{1} \tag{6.209}$$

$$= 0. \tag{6.210}$$

This shows that (6.191) holds, thus $[L(m),a(n)] = -na(m+n)$ for all $a \in \mathfrak{sl}_2(\mathbb{C})$ and $m,n \in \mathbb{Z}$.

Given $a \in \mathfrak{sl}_2(\mathbb{C})$ and $n \in \mathbb{Z}$, by (6.186) and the $\mathcal{D}$-bracket formula

$$[\mathcal{D}, Y(v, x)] = \frac{d}{dx} Y(v, x) \tag{6.211}$$

we have

$$[L(-1) - \mathcal{D}, a(n)] = 0 \tag{6.212}$$

as operators on $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)$, and

$$(L(-1) - \mathcal{D})\mathbf{1} = \omega_0 \mathbf{1} = \mathbf{1}_{-2}\mathbf{1} = 0. \tag{6.213}$$

Since $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)$ is generated from $\mathbf{1}$ by the universal enveloping algebra $U(\hat{\mathfrak{sl}}_2(\mathbb{C})$, it follows that $L(-1) = \mathcal{D}$ on $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)$. Also, $L(0) = d_{(0)}$ on $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)$ where $d_{(0)}$ is the weight operator defined by $d_{(0)}u = nu$ for $u \in V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)_{(n)}$ with $n \in \mathbb{Z}$. This shows that $L(0)v = nv$ for $v \in V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)_{(n)}$ where $n \geq 0$ and $L(-1) = \mathcal{D}$.

From the Virasoro algebra relations and $L(-1) = \mathcal{D}$ it is enough to show that

$$\omega_1 \omega = L(0)\omega = 2\omega, \tag{6.214}$$

$$\omega_3 \omega = L(2)\omega = \frac{dl}{2(l+h)}\mathbf{1} \tag{6.215}$$

$$\omega_n \omega = L(n-1)\omega = 0 \tag{6.216}$$

for $n = 2$ or $n \geq 4$. We have

$$L(0)\omega = \frac{1}{2(l+h)} \sum_{i=1}^{d} 2u^{(i)}(-1)^2\mathbf{1} = 2\omega, \tag{6.217}$$

$$L(2)\omega = \frac{1}{2(l+h)} \sum_{i=1}^{d} ([L(2), u^{(i)}(-1)]u^{(i)}(-1)\mathbf{1} + u^{(i)}(-1)[L(2), u^{(i)}(-1)]\mathbf{1}) \tag{6.218}$$

$$= \frac{1}{2(l+h)} \sum_{i=1}^{d} u^{(i)}(1)u^{(i)}(-1)\mathbf{1} \tag{6.219}$$

$$= \frac{1}{2(l+h)} \sum_{i=1}^{d} l\langle u^{(i)}, u^{(i)}\rangle\mathbf{1} = \frac{dl}{2(l+h)}\mathbf{1}, \tag{6.220}$$

$$L(1)\omega = \frac{1}{2(l+h)} \sum_{i=1}^{d} (u^{(i)}(0)u^{(i)}(-1)\mathbf{1} + u^{(i)}(-1)u^{(i)}(0)\mathbf{1}) = 0, \tag{6.221}$$

$$L(n)\omega = \frac{1}{2(l+h)} \sum_{i=1}^{d} (u^{(i)}(n-1)u^{(i)}(-1)\mathbf{1} + u^{(i)}(-1)u^{(i)}(n-1)\mathbf{1}) = 0 \tag{6.222}$$

for $n \geq 3$, completing the proof. □

Now we are ready to prove the main theorem.

Theorem 6.34. The vertex algebra $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)$ is a vertex operator algebra of central charge $dl/(l + h)$ with

$$\omega = \frac{1}{2(l+h)} \sum_{i=1}^{d} u^{(i)}(-1)u^{(i)}(-1)\mathbf{1} \in V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)_{(2)} \tag{6.223}$$

as the conformal vector. The $\mathbb{Z}$-grading is obtained by the $L(0)$ eigenvalues. Moreover, $\mathfrak{sl}_2(\mathbb{C}) = V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)_{(1)}$ generates $V_{\hat{\mathfrak{sl}}_2(\mathbb{C})}(l, 0)$ as a vertex algebra and

$$[L(m), a(n)] = -na(m + n) \tag{6.224}$$

for $a \in \mathfrak{sl}_2(\mathbb{C})$ and $m, n \in \mathbb{Z}$.

Proof. The proof follows immediately from Corollary 6.32 and Lemma 6.33. $\square$

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Curriculum Vitae

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