The Dirichlet Problem on Select Subsets of $\mathbb{R}^2$

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The Dirichlet Problem [5][6]

Definition

A real-valued function u on an open subset $\Omega \subseteq \mathbb{R}^n$ is harmonic if it is

1. twice continuously differentiable, and
2. the Laplacian of u, defined $\Delta u = \partial^2 u/\partial x_1^2 + ... + \partial^2 u/\partial x_n^2$, is 0 throughout Ω.

On some domain Ω, given data on the boundary, can we find a harmonic polynomial that matches the data on the boundary? Its main applications are in the physics of heat flow, electrostatics, and other fields.

General Solution in the Disk [5][6]

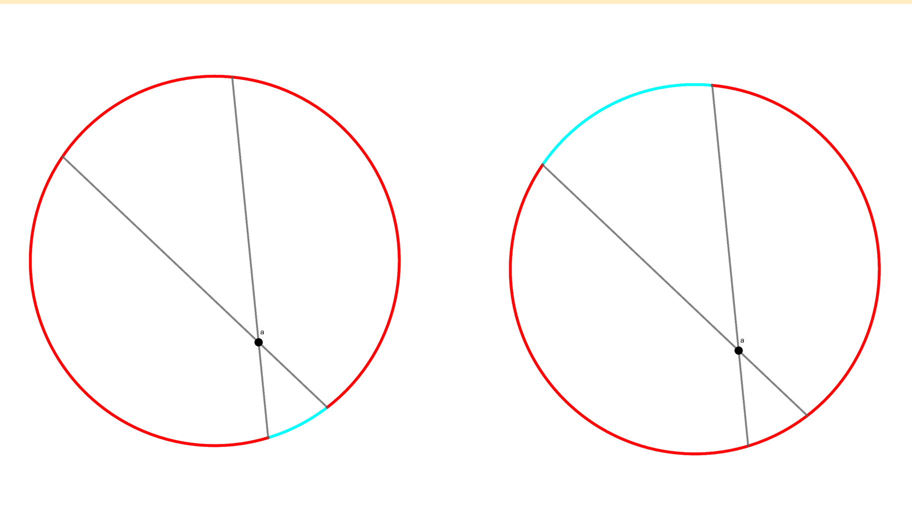
On disks centered at the origin, with boundary function T, the solution to the Dirichlet Problem is found in general by the Poisson Integral:

$$u(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\theta - \alpha)} \right] T(\theta)\, d\theta,$$

where any point in the disk $a = re^{i\alpha}, (r < R)$.

Schwarz Interpretation [5]

On disks in particular there is a visual interpretation of the Poisson Integral. Take the boundary data and reflect it across a given point a. A weighted average of the points on the reflected circle will equal the value of the solution to the Poisson Integral.



Conformal Maps

Let Ω be a domain in the plane such that there exists a conformal map $\varphi : \Omega \to D$ with $\varphi(\Omega) = D$, if we find a solution u to the Dirichlet problem to boundary data $R \circ \varphi^{-1}$ (on the unit disk), where R is the original boundary data function, $\varphi \circ u$ will still be harmonic, and a solution to the Dirichlet problem with the original parameters.

Complex analytic approach [1][6]

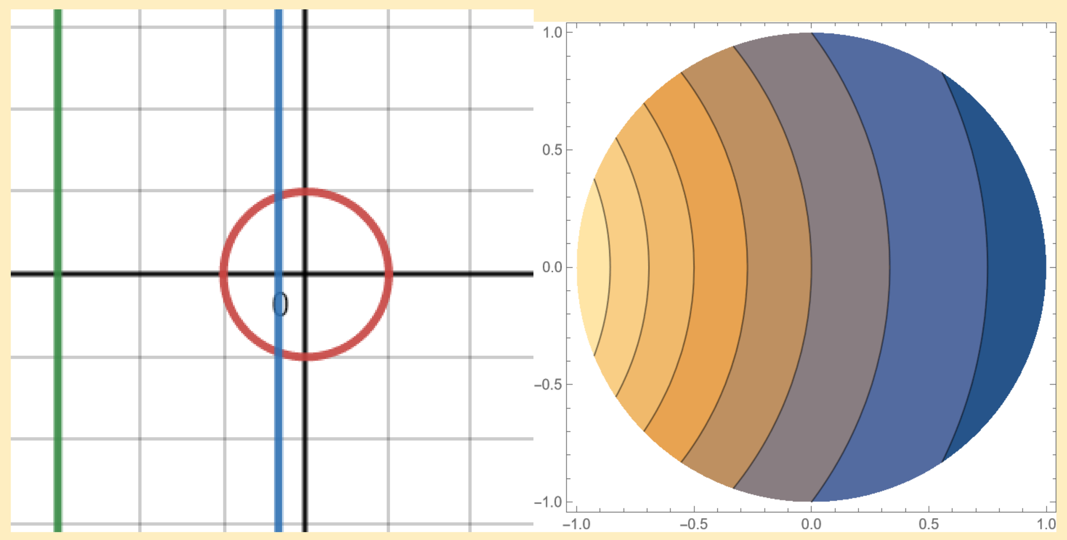
Consider the boundary data given by a rational function $R(x, y)$ on the boundary of the unit disk ∂D. Objective: It is known that the real part of an analytic function is harmonic. We wish to find $H(z)$ analytic on the disk D such that the real part of H equals R on the boundary ∂D. Use the change of coordinates $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$ to obtain a function of one complex variable

$$h(z) = R((z + 1/z)/2, (z - 1/z)/2i). \tag{1}$$

h is a rational function continuous on ∂D and equal to R on ∂D. We can decompose h into a sum of a polynomial and a rational function in z: $h(z) = p(z) + s(z)$. As a polynomial, $p(z)$ is already analytic. However $s(z)$ may have poles inside the disk, and so requires modification by reflecting the poles outside the disk. For each term $k_m(z) = a/(z - c)^{n_m}$ in $s(z)$ where $a, c \in \mathbb{C}$, $n \in \mathbb{Z}^+$, and $|c| < 1$, replace with the Kelvin transform

$$K(z) = \overline{k(1/\bar{z})} = \frac{\bar{a}z^n}{(1 - \bar{c}z)^n} \tag{2}$$

Note that the real parts of $K(z)$ and $k(z)$ are equal on the boundary, so the values of their real parts on the boundary stay the same. Define the function $H(z) = p(z) + S(z)$ where $S(z)$ is obtained from $s(z)$ by replacing each term $k(z)$ with $K(z)$ as described before. Our solution $u = \operatorname{Re} H$. One can show using this method that if R is a polynomial, so is u. If we let $R(x, y) = 1/(5 + 3x)$, then using this method gives us a function $h(z) = (1/4)/(3z + 1) + (3/4)/(z + 3)$. The below figure shows what happens to the pole inside the unit disk at $z = -1/3$ under the Kelvin transform: it gets moved outside to $z = -3$.



Fischer's Lemma and an algebraic solution [2][4]

The operator $L\colon \mathbb{P}[x_1, \dots, x_n] \longrightarrow \mathbb{P}[x_1, \dots, x_n]$ defined by $L(f) = \Delta(qf)$, where $q(x_1, \dots, x_n) = \sum_{k=1}^{n} x_k^2/r_k^2$ for $r_k > 0$. is a linear degree-preserving bijection from the space of real-valued polynomials of degree at most m to itself. This allows us to construct an algebraic solution to a Dirichlet problem over some region Ω whose boundary is given by q:

$$u = f - q \cdot L^{-1}(\Delta(f)) \tag{3}$$

Example

Suppose we have a Dirichlet problem over the unit disk with polynomial data given by $f = y^2$. When constructing the vector basis of $L\colon \mathbb{P}_2[x, y] \longrightarrow \mathbb{P}_2[x, y]$, we receive

$$[L] = \begin{bmatrix} 4 & 0 & 0 & -2 & 0 & -2 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14 & 0 & 2 \\ 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 2 & 0 & 14 \end{bmatrix}$$

Then we compute $[L^{-1}] = [L]^{-1}$ and apply the form in (3) to receive $u = y^2 - \frac{1}{2}(x^2 + y^2 - 1)$.

Homogeneous polynomial boundary data [3][6]

In the case of homogeneous polynomial data, we are able to directly compute a solution to the Dirichlet problem on a disk by way of harmonic decomposition. That is, every $p \in \mathcal{P}_m(\mathbb{R})$ can be uniquely written in the form

$$p = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} |x|^{2k} p_{m-2k} \tag{4}$$

where $p_k \in \mathcal{H}_k(\mathbb{R})$ for every k. It then follows that, if p is the boundary data function in a Dirichlet problem, then the solution to said Dirichlet problem, u, is given by

$$u = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} p_{m-2k} \tag{5}$$

Extending L to the case of rational boundary data

We are able to extend the use of L to the case of rational boundary data by introducing a restriction to rings of rational functions with a fixed denominator polynomial. However, we find that L, when applied to this restriction, is not necessarily one-to-one. In particular, for $L(P/Q) = \widetilde{P}/\widetilde{Q}$, it is the case that $\deg \widetilde{P} \leq \deg P + 2\deg Q$ and $\deg \widetilde{Q} = 3\deg Q$.

Restricting the domain of L to homogeneous polynomials

When restricting the domain of L to homogeneous polynomials, we are able to preserve the properties stated by Fischer's lemma. In particular, we find that

$$L\colon \bigoplus_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathcal{P}_{m-2k}(\mathbb{R}) \longrightarrow \bigoplus_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathcal{P}_{m-2k}(\mathbb{R}) \tag{6}$$

is a linear, degree-preserving bijection from $\bigoplus_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathcal{P}_{m-2k}(\mathbb{R})$ onto itself. Indeed, we can use this preservation of Fischer's lemma to show that $H_m(\mathbb{R})$ is invariant under L, since each vector in $\mathcal{H}_m(\mathbb{R})$ is an eigenvector with eigenvalue $4(m+1)$.

Discrete Poisson integral formula at the origin of the unit disk

In the polynomial data case, we can get a discrete sum from the Poisson integral formula for the value at the origin using the induction formulae for products of sines and cosines.

Vandermonde Matrices

The solution to the Dirichlet problem on the disk when the data is polynomial of degree m is the real part of an analytic function defined $u(z) = \frac{1}{2}\sum_{k=0}^{m} \left(c_k z^k + \bar{c}_k \bar{z}^k\right)$. We can determine the coefficients of $u(z)$ from its values at the roots of unity of order $(2m+1)$ by solving a linear system for which the coefficient matrix is a Vandermonde matrix with complex entries.

Further directions

- Can we get a discrete sum version of the Poisson integral formula for points other than the origin using interpolation?

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